STRUCTURE THEORY OF FINITE LIE CONFORMAL SUPERALGEBRAS

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Introduction

Lie conformal superalgebras encode the singular part of the operator product expansion of chiral fields in two-dimensional quantum field theory [K1]. On the other hand, they are closely connected to the notion of a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$, i.e. a Lie superalgebra $\mathfrak{g}$ spanned by the coefficients of a family $\mathcal{F}$ of mutually local formal distributions. Namely, to a Lie conformal superalgebra R one can associate a formal distribution Lie superalgebra $(\operatorname{Lie} R, R)$ which establishes an equivalence between the category of Lie conformal superalgebras and the category of equivalence classes of formal distribution Lie superalgebras obtained as quotients of $\operatorname{Lie} R$ by irregular ideals [K1].

The classification of finite simple Lie conformal superalgebras was completed in [FK]. The proof relies on the methods developed in [DK] for the classification of finite simple and semisimple Lie conformal algebras, and the classification of simple linearly compact Lie superalgebras [K4].

The main result of the present paper is the classification of finite semisimple Lie conformal superalgebras (Theorem 5.1). Unlike in [DK], we do not use the connection to formal distribution algebras in the proof of this theorem (which would require us to take care of numerous technical difficulties). We work instead entirely in the category of Lie conformal superalgebras. Our key result is the determination of finite differentiably simple Lie conformal superalgebras (Theorem 2.1). The proof of this result uses heavily the ideas of [B] and [C].

Given a finite Lie conformal superalgebra R, denote by $\operatorname{Rad} R$ the sum of all solvable ideals of R. Since the rank of a finite solvable Lie conformal (super)algebra is greater than the rank of its derived subalgebra [DK], we conclude that $\operatorname{Rad} R$ is the maximal solvable ideal, hence $R/\operatorname{Rad} R$ is a finite semisimple Lie conformal superalgebra. Thus in some sense the study of general finite Lie conformal superalgebras reduces to that of semisimple and solvable superalgebras.

In the Lie conformal algebra case there is a conformal analog of Lie's theorem, stating that any non-trivial finite irreducible module over a finite solvable Lie conformal algebra is free of rank 1 [DK]. However, a similar result in the "super" case is certainly false. We hope that we can develop a theory of finite irreducible modules over solvable Lie conformal superalgebras similar to that in the Lie superalgebra case [K2]. We make several observations to that end in the last section of this paper.

The paper is organized as follows: we define the main objects of our study and provide some general statements in Section 1. In Section 2 we establish the structure

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of differentiably simple Lie conformal superalgebras. Our proof is, in fact, quite general and the result is valid for non-Lie finite conformal superalgebras as well. We list finite simple Lie conformal superalgebras in Section 3 and describe their conformal derivations. Then in Section 4 we describe conformal derivations in the differentiably simple case and thus complete the classification of finite differentiably simple Lie conformal superalgebras. This allows us to describe the structure of finite semisimple Lie conformal superalgebras in Section 5. In Section 6 we classify simple physical Virasoro pairs and, as a consequence, obtain a classification of physical Lie conformal superalgebras which generalizes that of [K2]. Finally, in Section 7 we initiate the study of representations of finite solvable Lie conformal superalgebras.

1. Basic definitions and structures

1.1. **Formal distributions and conformal algebras.** Let $\mathfrak{g}$ be a Lie superalgebra. A $\mathfrak{g}$-valued *formal distribution* in one indeterminate z is a formal power series

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \quad a_n \in \mathfrak{g}.$$

The vector superspace of all formal distributions, $\mathfrak{g}[[z, z^{-1}]]$, has a natural structure of a $\mathbb{C}[\partial_z]$-module. We define

$$\operatorname{Res}_z a(z) = a_0.$$

Let $\mathfrak{g}$ be a Lie superalgebra, and let $a(z), b(z)$ be two $\mathfrak{g}$-valued formal distributions. They are called *local* if

$$(z-w)^N [a(z), b(w)] = 0 \quad \text{for} \quad N \gg 0.$$

Let $\mathfrak{g}$ be a Lie superalgebra, and let $\mathcal{F}$ be a family of $\mathfrak{g}$-valued mutually local formal distributions. The pair $(\mathfrak{g}, \mathcal{F})$ is called a *formal distribution Lie superalgebra* if $\mathfrak{g}$ is spanned by the coefficients of all formal distributions from $\mathcal{F}$.

The bracket of two local formal distributions is given by the formula

$$[a(z), b(w)] = \sum_j [a(w)_{(j)} b(w)] \partial_w^j \delta(z-w)/j!,$$

where $[a(w)_{(j)} b(w)] = \operatorname{Res}_z (z-w)^j [a(z), b(w)]$. Thus we get a family of operations ${}_{(n)}$, $n \in \mathbb{Z}_+$, on the space of formal distributions: $[a(z)_{(n)} b(z)]$ We define the λ-*bracket* on the space of formal distributions as the generating series of these operations [DK, K1]:

$$[a_\lambda b] = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!} (a_{(n)} b).$$

The properties of the λ-bracket lead to the following basic definition (see [K1, K2]):

A *Lie conformal superalgebra* R is a left $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map, called the λ-*bracket*,

$$R \otimes R \to \mathbb{C}[\lambda] \otimes R, \qquad a \otimes b \mapsto [a_\lambda b]$$

satisfying the following axioms ($a, b, c \in R$):

(sesquilinearity)	$[\partial a_\lambda b] = -\lambda[a_\lambda b],\ \ [a_\lambda \partial b] = (\partial+\lambda)[a_\lambda b]$,
(skew-commutativity)	$[b_\lambda a] = -(-1)^{p(a)p(b)}[a_{-\lambda-\partial} b]$,
(Jacobi identity)	$[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]]$.

(Here and further $p(a) \in \mathbb{Z}/2\mathbb{Z}$ stands for the parity of an element a.)

Similarly, one can define an *associative conformal superalgebra* by replacing the skew-commutativity and the Jacobi identity above with the following property:

$$\text{(associativity)} \qquad a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c.$$

As a shorthand, we call R *finite* if R is finitely generated as a $\mathbb{C}[\partial]$-module.

Below follow two useful constructions of Lie conformal superalgebras (more concrete examples will be discussed in Section 3):

Example 1.1. An associative conformal superalgebra can be endowed with a λ-bracket by putting

$$[a_\lambda b] = a_\lambda b - (-1)^{p(a)p(b)} b_{-\lambda-\partial} a.$$

Example 1.2. Let R be a Lie conformal superalgebra and let B be a commutative associative (ordinary) superalgebra. Then $R \otimes B$ carries a Lie conformal superalgebra structure defined as follows. The $\mathbb{C}[\partial]$-module structure is given by $\partial(r \otimes b) = (\partial r) \otimes b$ ($r \in R,\ b \in B$), and the λ-bracket by

$$[(r \otimes b)_\lambda (r' \otimes b')] = (-1)^{p(b)p(r')}[r_\lambda r'] \otimes (bb').$$

Notice that if R is finite and B is finite-dimensional, then $R \otimes B$ is also finite.

1.2. **Structural terminology.** We will denote by $\langle X_\lambda Y\rangle$ (or $\langle [X_\lambda Y]\rangle$, whenever appropriate) the $\mathbb{C}[\partial]$-module generated by elements of the form $x_{(n)}y$, where $x \in X, y \in Y$, and $n \in \mathbb{Z}_+$.

An *ideal* of a Lie conformal superalgebra R is a $\mathbb{C}[\partial]$-submodule I of R such that $\langle [R_\lambda I]\rangle \subset I$. An ideal I is *abelian* if $[I_\lambda I] = 0$, *central* if $[R_\lambda I] = 0$, and *nilpotent* if $\langle [\dots \langle [I_{\lambda_1} I]\rangle_{\lambda_2} \dots_{\lambda_k} I]\rangle = 0$ (in this case we sometimes write $I^{k+1} = 0$).

A Lie conformal superalgebra is *simple* if it is non-abelian and contains no ideals except for zero and itself.

The derived series of a Lie conformal superalgebra is built in the usual fashion: let $R' = \langle [R_\lambda R]\rangle$ and set $R^{(0)} = R$, $R^{(n+1)} = (R^{(n)})'$. Then a Lie conformal superalgebra R is *solvable* if $R^{(n)} = 0$ for some n. $\operatorname{Rad} R$ is the maximal solvable ideal of R (its existence is explained in the introduction).

A Lie conformal superalgebra is *semisimple* if it has no non-zero abelian ideals. Since the second last term of the derived series is an abelian ideal, this is equivalent to saying that R has no non-zero solvable ideals. Since the $\mathbb{C}[\partial]$-torsion is central [DK], a finite semisimple Lie conformal superalgebra is free as a $\mathbb{C}[\partial]$-module.

1.3. **Conformal modules.** Let V, W be $\mathbb{Z}_2$-graded left $\mathbb{C}[\partial]$-modules. We denote by $\operatorname{End}_{\mathbb{C}[\partial]} V$ the set of all $\mathbb{C}[\partial]$-linear endomorphisms of V. Notice that $\operatorname{End}_{\mathbb{C}[\partial]} V$ has a $\mathbb{C}[\partial]$-module structure given by

$$(\partial\varphi)(v) = \partial\varphi(v) \quad \text{for any} \quad v \in V.$$

A $\mathbb{C}$-linear map

$$\varphi \ : \ V \to \mathbb{C}[\lambda] \otimes_{\mathbb{C}} W$$

is called a *conformal linear map* if the following equation holds:

$$\varphi_\lambda(\partial v) = (\partial + \lambda)\varphi_\lambda v \quad \text{for any } v \in V.$$

The $\mathbb{C}$-vector space of all conformal linear maps from V to W is denoted by $\operatorname{Chom}(V, W)$. It has $\mathbb{C}[\partial]$-module structure if we set

$$(\partial\varphi)_\lambda(v) = -\lambda\varphi_\lambda v.$$

When $V = W$, we denote $\operatorname{Chom}(V, V)$ by $\operatorname{Cend} V$. When V is finite over $\mathbb{C}[\partial]$, $\operatorname{Cend} V$ becomes an associative conformal superalgebra with the λ-product

$$(a_\lambda b)_\mu v = a_\lambda(b_{\mu-\lambda} v), \quad a, b \in \operatorname{Cend} V.$$

The space $\operatorname{Cend} V$ endowed with a λ-bracket (see Remark 1.1) is denoted $\operatorname{gc} V$.

A *module* M over a Lie conformal superalgebra R is a $\mathbb{Z}_2$-graded left $\mathbb{C}[\partial]$-module M endowed with a $\mathbb{C}$-linear map

$$R \to \operatorname{gc} M.$$

Alternatively, one can define a module over R by providing a map $R \to \mathbb{C}[\lambda] \otimes \operatorname{End}_{\mathbb{C}} M$, $a \mapsto a^M_\lambda$ such that

$$(\partial a)^M_\lambda m = [\partial, a^M_\lambda]m = -\lambda a^M_\lambda m,$$
$$[a^M_\lambda, b^M_\mu]v = [a_\lambda b]^M_{\lambda+\mu} v.$$

An R-module M is *simple* if it has no nontrivial R-invariant $\mathbb{C}[\partial]$-submodules.

An *endomorphism* of an R-module M is a $\mathbb{C}[\partial]$-linear map $\phi \in \operatorname{End}_{\mathbb{C}[\partial]} M$ such that for any $a \in R$ and $v \in M$ we have

$$\phi(a_\mu v) = (-1)^{p(a)p(\phi)} a_\mu \phi(v).$$

1.4. **Derivations.** A *conformal derivation* of a Lie conformal superalgebra R is a conformal endomorphism ϕ of R such that for any homogeneous $x, y \in R$

$$\phi_\lambda[x_\mu y] = [(\phi_\lambda x)_{\lambda+\mu} y] + (-1)^{p(x)p(\phi)}[x_\mu(\phi_\lambda y)].$$

We denote by $\operatorname{Cder} R$ (resp. $\operatorname{Cinder} R$) the space of all conformal derivations of R (resp. of all *inner conformal derivations*, i.e. conformal derivations of the form $\operatorname{ad} a$, $a \in R$, $(\operatorname{ad} a)_\lambda b = [a_\lambda b]$ for $b \in R$). Clearly $\operatorname{Cder} R$ is a subalgebra of $\operatorname{gc} R$.

Let D be a set of conformal derivations of R. An ideal I of R is *D-stable* if $\phi I \subseteq I$ for all $\phi \in D$. Here (and below) we use the shorthand ϕI for $\langle \phi_\lambda I \rangle$.

A Lie conformal superalgebra is *D-differentiably simple* if it contains no proper D-stable ideals. A Lie conformal superalgebra is *differentiably simple* if it is D-differentiably simple with respect to some D.

We will also work with ordinary derivations of conformal algebras. An *ordinary derivation* of a Lie conformal superalgebra R is a $\mathbb{C}[\partial]$-linear endomorphism d of R such that for any homogeneous $x, y \in R$,

$$d([x_\mu y]) = [d(x)_\mu y] + (-1)^{p(x)p(d)}[x_\mu d(y)].$$

We denote the space of all ordinary derivations of R by $\operatorname{Der} R$. Remark that a conformal derivation ϕ gives rise to an ordinary derivation $\phi_{(0)}$. In particular, every element $a \in R$ gives rise to an ordinary derivation $\operatorname{ad} a_{(0)}$. We call such derivations *inner* and denote their space as $\operatorname{Inder} R$.

Remark 1.3. The operator ∂ always acts as an ordinary derivation of a conformal algebra. In some cases it is inner (e.g. when the algebra possesses a Virasoro element, see Example 3.2).

1.5. **Minimal ideals.** Here we collect some facts about minimal ideals of finite Lie conformal superalgebras.

Lemma 1.4. *Let R be a finite Lie conformal superalgebra and J an ideal of R which contains no nonzero central elements. Then J contains a minimal ideal of R.*

Proof. Let I be a minimal rank ideal of R contained in J. Let $I_0 = \cap_i K_i$ be the intersection of all non-zero ideals of R contained in I. Any ideal K_i has the same rank as I, hence I/K_i is a torsion $\mathbb{C}[\partial]$-module and by [DK, Proposition 3.2], R acts trivially on it. This means that for any i we have $\langle [R_\lambda I]\rangle \subseteq K_i$. Therefore $\langle [R_\lambda I]\rangle \subseteq I_0$. Note that $\langle [R_\lambda I]\rangle \neq 0$ because otherwise I would be a central ideal of R. Hence $I_0 \neq 0$ and clearly I_0 is a minimal ideal of R. □

Lemma 1.5. *Let M be a nonabelian ideal in a finite Lie conformal superalgebra R. Then*

1. *M is a minimal ideal if and only if M is* $\operatorname{ad} R$*-simple.*
2. *If M is a minimal ideal, then it is differentiably simple.*
3. *If M is a minimal ideal, then it is* $\operatorname{Cder} R$*-invariant and* $\operatorname{Cder} R$*-simple.*
4. *If M is a minimal ideal, then either M is simple or M contains a minimal ideal I of M which is abelian.*

Proof. (1) is immediate. (2) follows from (1) and the fact that $\operatorname{ad} R \subseteq \operatorname{Cder} M$. As for (3), we remark that the minimality of M implies that $\langle [M_\lambda M]\rangle = M$, hence $\langle \phi_\lambda(M)\rangle \subseteq \langle [(\phi_\mu M)_\lambda M]\rangle \subseteq M$ for any $\phi \in \operatorname{Cder} R$, i.e. M is a $\operatorname{Cder} R$-invariant ideal of R. Now, let J be a nonzero $\operatorname{Cder} R$-invariant ideal of R, which is contained in M. The minimality of M implies that $J = M$. In order to prove (4), notice that by (2) M is differentiably simple. Suppose M is not simple. The center of M is a differential ideal of M, hence it is zero and Lemma 1.4 provides a minimal ideal I in M. Suppose that I is not abelian. Then $\langle [I_\lambda I]\rangle = I$ and I is a $\operatorname{Cder} M$-invariant ideal in M which is differentiably simple. Hence $I = M$ i.e. M is a minimal ideal in itself. We conclude that M is simple. The contradiction proves that I is abelian. □

Note that by Lemma 1.5(3), any minimal ideal in a differentiably simple but non-simple finite Lie conformal superalgebra is abelian.

2. Differentiably simple conformal superalgebras

In this section we prove the following

Theorem 2.1. *Let R be a finite non-abelian differentiably simple Lie conformal superalgebra. Then $R \simeq S \otimes \wedge(n)$ for a simple Lie conformal superalgebra S.*

Remark 2.2. Our proof also works for any finite non-abelian differentiably simple conformal superalgebra (i.e. the one for which only sesquilinearity holds) but we do not require the result in this generality.

2.1. **Centroid.** Let R be a Lie conformal superalgebra and M an R-module. For $x \in R$, let L_x be an element of $\operatorname{Cend} R$ such that $(L_x)_\lambda y = x_\lambda y$ for any $y \in M$.

Definition 2.3. The *centroid* $C(M)$ of a module M over a Lie conformal superalgebra R is the subalgebra of the associative superalgebra $\operatorname{End}_{\mathbb{C}} M$ that consists of elements (super)commuting with $(L_x)_{(n)}$ for all $x \in R$, $n \in \mathbb{Z}_+$ and the action of ∂.

Remark 2.4. By definition $C(M)$ is a subalgebra of the associative superalgebra $\operatorname{End}_{\mathbb{C}[\partial]} M$.

Remark 2.5. We We show in the proof of Lemma 2.6 below that for $a \in C(R) \subset \operatorname{End}_{\mathbb{C}[\partial]} R$, $a(x_\lambda y) = (ax)_\lambda y = (-1)^{p(a)p(x)}(x_\lambda ay)$. These conditions can be taken as the definition of the centroid of a conformal superalgebra R. (Note that here R is not necessarily Lie.)

Lemma 2.6. *If $R = \langle [R_\lambda R] \rangle$, then $C(R)$ is (super)commutative.*

Proof. The lemma follows from the equalities $a[x_\lambda y] = (-1)^{p(a)p(x)}[x_\lambda ay]$ and $a[x_\lambda y] = [ax_\lambda y]$ for any $a \in C(R), x, y \in R$. The first equality follows directly from the definition; the second is deduced from the first:

$$\begin{aligned} a[x_\lambda y] =& - (-1)^{p(x)p(y)} a[y_{-\lambda-\partial} x] = -(-1)^{p(x)p(y)+p(a)p(y)}[y_{-\lambda-\partial} ax] \\ =& (-1)^{p(x)p(y)+p(a)p(y)+(p(a)+p(x))p(y)}[ax_\lambda y] = [ax_\lambda y]. \end{aligned}$$

Then for any $a, b \in C(R)$ and any $x, y \in R$, $(ab)[x_\lambda y] = (-1)^{p(b)p(x)}[ax_\lambda by]$ and $(ba)[x_\lambda y] = (-1)^{(p(a)+p(x))p(b)}[ax_\lambda by]$, implying that a and b (super)commute. $\square$

Since $\langle [R_\lambda R] \rangle$ is a differentiably stable ideal, it follows that $C(R)$ is (super)commutative for a differentiably simple R.

Lemma 2.7. *For a homogeneous (i.e. even or odd) $a \in C(R)$, $\ker a$ and $\operatorname{im} a$ are ideals of R.*

Proof. Clear. $\square$

We also have a version of the Schur Lemma:

Lemma 2.8. *Let M be a countable dimensional simple module over a Lie conformal superalgebra R. Then either $C(M) = \mathbb{C}1_M$ or $\dim M_{\bar{0}} = \dim M_{\bar{1}}$ and*

$$C(M) = \mathbb{C}1_M \oplus \mathbb{C}U,$$

where U is an odd operator such that $U^2 = 1_M$.

Proof. The proof follows the classical line of argument. Let $a \in C(M)$ be a non-zero even operator. The fact that M is simple implies that a is invertible. Suppose a is not a scalar. Since $\mathbb{C}$ is algebraically closed, a cannot be an algebraic element in $C(M)$. The field of rational functions in a over $\mathbb{C}$ is contained in $C(M)$, hence $C(M)$ has dimension greater than countable. On the other hand, let us fix a nonzero $x \in M$. The map $C(M) \to M$ sending a to ax is injective because $C(M)$ is a division ring. However, M is countable dimensional. The contradiction proves that $a = c1_M$ for some $c \in \mathbb{C}$.

Let $a \in C(R)$ be a non-zero odd operator. Then b is invertible (this can only happen if $\dim M_{\bar{0}} = \dim M_{\bar{1}}$). Furthermore, a^2 is an even operator, hence a scalar. Suppose a_1, a_2 are two non-zero odd operators in $C(M)$. Then $(a_1 - ca_2)^2 \in \mathbb{C}1_M$ for any $c \in \mathbb{C}$. Assume that for c and c', $(a_1 - ca_2)^2 \neq 0$ and $(a_1 - c'a_2)^2 \neq 0$. Then $(a_1 - ca_2)^2 + r^2(a_1 - c'a_2)^2 = 0$ for some $0 \neq r \in \mathbb{C}$. We obtain two non-zero odd operators b_1 and b_2 such that $b_1^2 + b_2^2 = 0$. By taking the square of $b_1 b_2 b_1^{-1} b_2^{-1}$ one easily shows that $b_1 b_2 = \pm b_1 b_2$. Thus either $(b_1 + b_2)^2 = 0$ or $(b_1 + ib_2)(b_1 - ib_2) = 0$ and it follows that a_1 is proportional to a_2. $\square$

Corollary 2.9. *If R is a countably dimensional simple Lie conformal superalgebra, then $C(R) = \mathbb{C}1_R$.*

Proof. Assume $C(R) = \mathbb{C}1_R \oplus \mathbb{C}U$, U^2=1. Let $a = (1+U)/2$ and $\bar{a} = (1-U)/2$. R splits as $aR \oplus \bar{a}R$ and it is easy to see that $\ker a = \bar{a}R$. Using the equality $b[x_\lambda y] = [bx_\lambda y]$, we see immediately that $\ker a$ and $\operatorname{im} a$ are (non-homogeneous) ideals of R. On the other hand, $[x_\lambda ay] = \bar{a}[x_\lambda y]$ for x odd, hence $aR \cap \bar{a}R \neq \{0\}$, a contradiction. □

Remark 2.10. Lemma 2.8 and Corollary 2.9 hold for ordinary Lie superalgebras (with the same proof).

2.2. **Conformal Derivations and the centroid.** Let $\phi \in \operatorname{Cder} R$, $\gamma \in \mathbb{C}$, and $n \in \mathbb{Z}_+$. We introduce the following operators acting on $\operatorname{End} R$: $\operatorname{ad}\phi_\gamma = [\phi_\gamma, \bullet]$ and $\operatorname{ad}\phi_{(n)} = [\phi_{(n)}, \bullet]$.

Lemma 2.11. *Let $a \in C(R)$, $\phi \in \operatorname{Cder} R$.*

1) For any $\gamma \in \mathbb{C}$, $\operatorname{ad}\phi_\gamma(a) \in C(R)$. Moreover, $\operatorname{ad}\phi_\gamma$ is a derivation of $C(R)$.

2) For any $n \in \mathbb{Z}_+$, $\operatorname{ad}\phi_{(n)}(a) \in C(R)$. Moreover, $\operatorname{ad}\phi_{(n)}$ is a derivation of $C(R)$.

Proof. Direct computation. □

Lemma 2.12. *If $a \in C(R)$ and $\phi \in CderR$, then $a\phi \in CderR$. Also $p(a\phi) = p(a) + p(\phi)$.*

Proof. Direct computation. □

2.3. **Constructing a chain of ideals.** Let R be a finite non-abelian differentiably simple Lie conformal superalgebra. Remark that the center of R, $\{x \mid [x_\lambda R] = 0\}$ is a differentiably stable ideal, hence it is zero and by Lemma 1.4, R contains a minimal ideal I. Let D be a set of homogeneous conformal derivations of R. Our ultimate goal is to construct with the use of D a certain finite chain of ideals that starts with I (cf. [B]).

Assume now that we have constructed a chain of ideals $I = I_1 \subset I_2 \subset \cdots \subset I_q \neq R$ such that for all $2 \le j \le q$, $I_j/I_{j-1} \simeq I$ as R-modules. Let $\phi \in D$ be a homogeneous conformal derivation such that $\phi I_q \not\subset I_q$. We are going to construct an ideal $I_{q+1} \supset I_q$ such that $I_{q+1}/I_q \simeq I$ as R-modules.

Remark, first that for any ideal J, a homogeneous conformal derivation ϕ induces a map $\bar\phi \in \operatorname{Chom}(J, R/J)$, $\bar\phi_\lambda x = \phi_\lambda x + \mathbb{C}[\lambda]\otimes J$. Moreover, by definition, for every $y \in R$, the following equalities holds in the R-module R/J:

$$\bar\phi_\lambda(y_\mu x) = (-1)^{p(\phi)p(y)}(y_\mu(\bar\phi_\lambda x));$$
$$\partial(\bar\phi_\lambda x) = \bar\phi_\lambda(\partial x) - \lambda\bar\phi_\lambda x.$$

It follows that for any $\gamma \in \mathbb{C}$, $\ker\bar\phi_\gamma$ is a homogeneous ideal of R and $\operatorname{im}\bar\phi_\gamma$ is an R-submodule of R/J.

Suppose now that we have constructed a chain such as above: $I_1 \subset \cdots \subset I_q \neq R$. Let j be minimal such that $\phi I_j \not\subset I_q$. Take $J = I_q$ and consider the map $\bar\phi \in \operatorname{Chom}(I_q, R/I_q)$ constructed as above. Restrict this map to I_j. By construction $\phi(I_{j-1}) \subseteq I_q$, so $I_{j-1} \subseteq \ker\bar\phi_\gamma$ for any $\gamma \in \mathbb{C}$. Thus we have a family of maps $\bar\phi_\gamma : I_j/I_{j-1} \to R/I_q$.

We will show that there exists γ such that $\ker\bar\phi_\gamma$ is zero. Indeed, if for some $\gamma \in \mathbb{C}$ there exists $x + I_{j-1} \in I_j/I_{j-1}$ such that $\bar\phi_\gamma x = 0$, then $\bar\phi_\gamma(I_j/I_{j-1}) = 0$ (it is a simple R-module by construction). Hence, if there exists such an $x + I_{j-1}$ for every γ, then $\bar\phi_\lambda(I_j/I_{j-1}) = 0$ and, consequently, $\phi I_j \subset I_q$.

Hence we can find $\gamma \in \mathbb{C}$ with $\ker \bar{\phi}_\gamma = 0$. Then we can define I_{q+1} as an ideal such that $I_{q+1}/I_q = \operatorname{im} \bar{\phi}_\gamma$.

Remark 2.13. If we start constructing a chain of ideals such as above and keep the same $\phi \in D$, we would at some point obtain I_q such that $\phi I \subset I_q$. Indeed, pick a basis $x_1, \dots, x_n$ of I over $\mathbb{C}[\partial]$ and let d be the maximal degree in λ of all $\phi_\lambda x_i$. At every step of the above construction, we produce an ideal I_k and $\gamma_k \in \mathbb{C}$ such that $\phi_{\gamma_k} I_1 \subset I_k$. It is clear that $\gamma_1, \dots, \gamma_d$ are pairwise distinct, thus $\phi_\lambda I_1 \subset I_d$ as required.

2.4. **Constructing the maximal ideal.** Now let D be a finite collection of homogeneous conformal derivations, $D = \{\phi_1, \dots, \phi_m\}$, such that I is not D-stable. Let i_1 be the least index for which $\phi_{i_1} I \not\subset I$. We apply the algorithm from the previous subsection. Namely, using ϕ_{i_1} we can construct a chain of ideals $I = I_1 \subset \cdots \subset I_{r_1}$, where $\phi_{i_1} I \subset I_{r_1}$. Then use ϕ_{i_2} (with the minimal i_2) such that $\phi_{i_2} I_{r_1} \not\subset I_{r_1}$ to extend the chain to I_{r_2}, where $\phi_{i_2} I_{r_1} \subset I_{r_2}$, etc.

Either at some point we obtain $I_l = R$ or we obtain a proper ideal I_q such that $\phi_m \dots \phi_1 I \subseteq I_q$.

Suppose now that for every collection D only the second case occurs. Since R is differentiably simple, there exists a homogeneous conformal derivation ϕ such that $\phi I_q \not\subset I_q$ and, using ϕ, we can extend our chain to I_{q+1}. For any $x \in I$, let $y = \phi_{m\,(n_m)}(\dots \phi_{1\,(n_1)} x) \dots)$. Hence $y \in I_q$ and $y_\lambda I_1 \simeq y_\lambda (I_{q+1}/I_q) = 0$. But every $y \in R$ can be expressed as a $\mathbb{C}[\partial]$-linear combination of elements in the above form (for some collection $D = \{\phi_1, \dots, \phi_m\}$). Hence, $[R_\lambda I] = 0$, i.e. R has a non-trivial center, a contradiction.

Therefore, there exists a finite chain $I = I_1 \subset \cdots \subset I_l = R$ such that for every $2 \le j \le l$, $I_j/I_{j-1} \simeq I$ as R-modules. Denote I_{l-1} as N. Then $R/N \simeq I$ and N is maximal.

Since $N_\lambda(R/N) = 0$, $\langle [N_\lambda I_j] \rangle \subset I_{j-1}$ for any j. Thus $N^l = 0$ and N is nilpotent.

Suppose that there exists another maximal ideal N' of R. Then $N + N' = R$ and R/N' is nilpotent and simple, a contradiction (as $R = \langle [R_\lambda R] \rangle$). We arrive at

Proposition 2.14. *Let R be a finite non-abelian differentiably simple conformal Lie superalgebra and let I be its minimal ideal. Then R possesses a chain of ideals $I = I_1 \subset I_2 \subset \cdots \subset N \subset R$, where N is a unique maximal ideal of R. Each factor I_j/I_{j-1} is isomorphic to I as R-modules. Moreover, N is nilpotent and $R/N \simeq I$.*

Actually, above we only used that I is finite, this the proof implies the following aside corollary.

Corollary 2.15. *If R is a non-abelian differentiably simple Lie conformal superalgebra with a finite minimal ideal, then R itself is finite.*

Remark 2.16. Uniqueness of N implies that all minimal ideals of R are R-isomorphic.

2.5. **Centroid structure.** Here we construct an embedding $\sigma : C(R/N) \to C(R)$. For $a \in C(R/N)$, define $\sigma(a)$ as the composition of maps

$$R \twoheadrightarrow R/N \xrightarrow{a} R/N \xrightarrow{\rho} I \hookrightarrow R$$

(here the isomorphism $\rho : R/N \xrightarrow{\sim} I$ is the one constructed in the proof of Proposition 2.14).

Clearly, σ is an embedding; moreover, $\operatorname{im} \sigma(a) = I$ for all a.

For an arbitrary R-isomorphism $\theta : R/N \xrightarrow{\sim} I$, consider the map $\beta : R/N \xrightarrow{\theta} I \xrightarrow{\rho^{-1}} R/N$. Since β commutes with the action of R and ∂, we see that $\sigma(\beta) = \theta$.

Let $I = I_1 \subset I_2 \subset \ldots \subset N \subset R$ be the chain described in Proposition 2.14.

Lemma 2.17. *There exists a family of monomorphisms $\sigma_q : C(R/N) \to C(R)$, $1 \le q \le l$, such that for every $a \in C(R/N)$ $\sigma_q(a)R + I_{q-1} = I_q$ and $\sigma_q(a)N \subseteq I_{q-1}$. Moreover, if $\theta : R/N \to I_q/I_{q-1}$ is an R-isomorphism, then there exists $b \in C(R/N)$ such that θ is induced by $\sigma_q(b)$.*

Proof. We put $\sigma_1 = \sigma$ constructed above. Then denote by d_q the map ϕ_γ that we used to construct I_{q+1} and let I_j be the ideal used in that construction. Put $\sigma_{q+1} = [d_q, \sigma_j]$ (a super-bracket). Modulo I_{j-1}, $\sigma_j(a)R = I_j$ for any nonzero a, hence modulo I_q, $d_q\sigma_j(a)R = I_{q+1}$. Since $\sigma_j(a)d_qR \subseteq I_j \subset I_q$, $\sigma_{q+1}R + I_q = I_{q+1}$ as required. Similarly, $\sigma_{q+1}N \subseteq I_q$.

Given $\theta : R/N \to I_{q+1}/I_q$, we can extend it to a map $R/N \to I_j/I_{j-1}$ that is induced by $\sigma_j(b)$ for some b. Thus $\sigma_{q+1}(b)$ induces θ. ☐

Using the maps σ_q constructed in the above lemma, we have

Lemma 2.18. *The map $\widetilde{\sigma} = \oplus_{q=1}^{l}\sigma_q$ is an isomorphism $\bigoplus_{l \text{ copies}} C(R/N) \xrightarrow{\sim} C(R)$. Also, $J = \sigma_1(C(R/N))$ is a minimal ideal of $C(R)$.*

Proof. Let $a \in C(R)$ and q be minimal such that $aR \subseteq I_q$. Combined with a projection $R \to R/I_{q-1}$, a induces a map $R \to I_q/I_{q-1}$. Its kernel is a maximal ideal, i.e. N. Hence a induces an isomorphism $R/N \to I_q/I_{q-1}$. Then there exists $b \in C(R/N)$ such that $(a - \sigma_q(b))R \subseteq I_{q-1}$. By induction, $\widetilde{\sigma}$ is onto.

Conversely, if $\widetilde{\sigma}(\oplus_1^l a_i) = 0$, then $\sigma_l(a_l)R \subseteq N$. Thus $a_l = 0$. By induction, all $a_i = 0$.

The set $J = \sigma_1(C(R/N)) = \{a \mid aR \subset I\}$ is obviously an ideal of $C(R)$. Let $a', a'' \in J$. Then they induce isomorphisms $\theta', \theta'' : R/N \to I$. Then $a' = ba''$, where $b \in C$ induces $\theta'^{-1}\theta''$. ☐

Remark 2.19. We actually obtain a chain of ideals $J = J_1 \subset \cdots \subset J$ with properties similar to that of $I_1 \subset \cdots \subset R$.

Corollary 2.20. *$C(R)$ is finite-dimensional.*

Proof. Follows from Lemmas 2.8 and 2.18. ☐

Denote by D the subset of $\operatorname{Der} C(R)$ that consists of derivations $\operatorname{ad}\phi_\gamma$, $\phi \in \operatorname{Cder} R$, $\gamma \in \mathbb{C}$.

Lemma 2.21. *$C(R)$ is D-differentiably simple.*

Proof. Let H be a D-stable ideal of $C(R)$. Then $\phi_\gamma HR \subseteq HR$ for any $\phi \in \operatorname{Cder} R$, $\gamma \in \mathbb{C}$. Hence, $\phi(HR) \subseteq HR$ and $HR = R$. There exists $h \in H$ such that $hR + N = R$ (as N is maximal). Thus $0 \ne J_1 h \subseteq H \cap J_1$, i.e. $J_1 \subseteq H$. The construction of maps σ_q and $\widetilde{\sigma}$ in Lemmas 2.17 and 2.18 implies that $C \subseteq H$. ☐

Corollary 2.22. *Let $D' = \{\operatorname{ad}\phi_{(n)}\}$. Then $C(R)$ is D'-differentiably simple.*

Proof. A D'-stable ideal of $C(R)$ is D-stable by the definition of ϕ_γ. ☐

Since $C(R)$ is differentiably simple and finite-dimensional, we obtain from [C] the following

Proposition 2.23. *Let $C(R)$ be the centroid of a finite non-abelian differentiably simple conformal Lie superalgebra. Then $C(R)$ is a Grassmann superalgebra $\wedge(r)$.*

2.6. **More on differential simplicity.** At this point the non-conformal argument for a Lie superalgebra proceeds as follows: first, establish that $C(L)$ is $\operatorname{ad} d$-differentiably simple for just one conformal derivation $d \in \operatorname{Der} L$ (with certain additional conditions on d in the super case) and then conclude that L is d-differentiably simple.

In order for this to work for a Lie conformal superalgebra R as well, we need conformal analog of d of the form $\operatorname{ad}\phi_\gamma$ for some γ. We need to tweak the proof of Theorem 5.1 in [C].

We restate the theorem itself first. Let C be a Grassmann algebra over an algebraically closed field of characteristic 0. Let C be D-differentiably simple with respect to a homogeneous set of derivations D which is both a subalgebra and a left C-module. Then there exists $d \in D$ such that C is d-differentiably simple and for the corresponding chain $J = J_1 \subset \cdots \subset C$, the map $\bar{d} : J_q \to C/J_q$ is homogeneous (i.e. $dx \equiv d_\epsilon x \mod J_q$ for all $x \in J_q$, $\epsilon = \bar{0}, \bar{1}$).

The reason for the last condition is simple: in order to build a chain of ideals we need to use a homogeneous derivation at every step. In this case we say that d is *homogeneous at every step.*

The major steps of the proof are: find a homogeneous nilpotent $m \in C = \wedge(r)$ and a derivation d_1 such that $d_1(m) = 1$. Then $D = D_0 \oplus Cd_1$, where $D_0 = \{d - d_1(m)d_1 \mid d \in D\}$ (in particular, D_0 acts as 0 on m). View C as a $\langle C, D_0\rangle$-module and use d_1 to construct the appropriate chain of ideals. The minimal ideal in this chain is isomorphic to $\wedge(r_1)$, $r_1 < r$, so we can use $d_0 \in D_0$ to refine it into the chain of ideals of C.

It follows from the proof that the requirement of D being an algebra is superfluous, thus in order to apply the construction of d_0 and d_1 to the set D' constructed in Corollary 2.22, we have to show that it is a C-module. It suffices to show that $d = \operatorname{ad}\phi_{(n)} + \operatorname{ad}\psi_{(m)} \in D'$ for $m \neq n$. Let $m < n$. Then $d = (\operatorname{ad}\phi + (-1)^{n-m}\frac{m!}{n!}\partial^{n-m}\psi)_{(n)}$.

In the same vein we can assume that d_0 and d_1 are of the form $\operatorname{ad}\phi_{(n)}$ for the same n. Hence, we can strengthen the theorem as

Lemma 2.24. *There exist $\phi \in \operatorname{Cder} R$ and n such that $C(R) = \wedge(r)$ is $\operatorname{ad}\phi_{(n)}$-differentiably simple and $\operatorname{ad}\phi_{(n)}$ is homogeneous at every step.*

Lemma 2.25. *There exist $\phi \in \operatorname{Cder} R$ and $\gamma \in \mathbb{C}$ such that $C(R)$ is $\operatorname{ad}\phi_\gamma$-differentiably simple.*

Proof. Let ϕ and n be as in the previous lemma. Assume that for each $\gamma \in \mathbb{C}$, there exists an ideal J_γ that is $\operatorname{ad}\phi_\gamma$-stable. Since $C(R) \simeq \wedge(n)$ has a unique minimal ideal, $\cap J_\gamma$ is non-empty. Thus, there exists a proper ideal J that is $\operatorname{ad}\phi_\gamma$-stable for all γ. Hence, $\operatorname{ad}\phi_\lambda$ maps J into $J[\lambda]$ and J is $\operatorname{ad}\phi_{(n)}$-stable. □

Clearly, $\operatorname{ad}\phi_\gamma$ is not necessarily homogeneous at every step. Let D be the $C(R)$-module generated by the homogeneous components of $\operatorname{ad}\phi_\gamma$. Notice that since $\operatorname{ad}\chi_\gamma + \operatorname{ad}\phi_\gamma = \operatorname{ad}(\chi + \psi)_\gamma$, every element of D arises from a conformal derivation of R.

Since $C(R)$ is D-differentiably simple, we can apply the previous argument to obtain

Lemma 2.26. *There exist ψ and γ such that $C(R)$ is $\operatorname{ad}\psi_\gamma$-differentiably simple. Moreover, $\operatorname{ad}\psi_\gamma$ is homogeneous at every step.*

Let ψ and γ be as in Lemma 2.26. We claim that R contains no ψ-stable homogeneous ideals. Indeed, let M be such an ideal. We can always assume that $I \subset M$ (e.g. let I be a minimal ideal contained in M). Moreover, $M \subset N$. Let $H = \{a \,|\, aR \subset M\}$, a proper homogeneous ideal of $C(R)$. A direct calculation shows that H is $\operatorname{ad}\psi_\gamma$-stable, a contradiction.

Thus R is ψ-stable.

We need to show that ψ_γ acts homogeneously at every step, i.e. that the map $\bar{\psi}_\gamma : I_q \to R/I_q$ is homogeneous. This will allow us to build the chain $\{I_q\}$. It suffices to show that one of the homogeneous components of $\bar{\psi}_\gamma$ acts by zero. So, let ϕ and β be such that ϕ_β is homogeneous and $\operatorname{ad}\phi_\beta : J_q \to C/J_q$ acts by zero. (We assume here that we have already built the chain $\{J_q\}$ and that the chain of ideals of R has been constructed up to I_q. We also assume that $\phi_\beta I_{q-1} \subset I_q$.) There exist $c \in J_q$ and $y \in R$ such that cy generates I_q/I_{q-1} over R. Then $\psi_\beta cy = [\psi_\beta, c]y \pm c\psi_\beta y \in I_q$. It follows that $\psi_\beta R_\lambda cy \subset I_q[\lambda]$. Summing up, we have

Lemma 2.27. *Let R be a finite non-abelian differentiably simple Lie conformal algebra. Then there exist a conformal derivation ψ and $\gamma \in \mathbb{C}$ such that R is ψ_γ-differentiably simple. Moreover, using ψ_γ we can construct a chain of homogeneous ideals $I = I_1 \subset \cdots \subset N \subset R$.*

2.7. **Splitting R.** Let R be a finite non-abelian differentiably simple Lie conformal algebra. Let ψ and γ be as in the Lemma 2.27. Let $S = \{x \,|\, \psi_\gamma x \in I\}$.

Lemma 2.28. *$R = S \oplus N$ as $\mathbb{C}[\partial]$-submodules and S is a conformal subalgebra of R.*

Proof. A direct computation shows that S is closed with respect to the ∂-action and the λ-bracket inherited from R.

Let $0 \neq x \in S \cap N$. Then $x \in I_q \backslash I_{q-1}$ for some q. Hence, $\psi_\gamma x \notin I_q$, a contradiction, and $S \cap N = 0$.

To show that $R = S + N$, observe that $I_q \subseteq I + \psi_\gamma N$ for all q. Indeed, by induction $I_{q-1} \subseteq I + \psi_\gamma N$ and $\psi_\gamma I_{q-1} \subset \psi_\gamma N$. Hence $I_q = I_{q-1} + \psi_\gamma I_{q-1} \subseteq I + \psi_\gamma N$. In particular $R \subseteq I + \psi_\gamma N$. Thus for every $x \in R$, $\psi_\gamma x = \psi_\gamma y \mod I$ for some $y \in N$ and $x - y \in S$. ☐

Corollary 2.29. *S is a finite simple Lie conformal superalgebra.*

Proof. $S \simeq R/N$ as conformal superalgebras. ☐

Let $f : S \to R$ be the natural embedding. Define the map $F : \wedge(r) \otimes S \to R$ by setting

$$F(c \otimes x) = (-1)^{p(c)p(x)} c f(x), \qquad c \in \wedge(r), x \in R$$

for homogeneous elements.

Recall that we have constructed the chain $\{J_q\}$ of ideals of $C(R)$. In particular, $J_q = \sigma_1(C(S)) + \cdots + \sigma_q(C(S))$. Denote $c_q = \sigma_q(1)$ (it is homogeneous by definition of σ_q). Clearly $F(c_1 \otimes S) = I$. By induction $I_q + F(c_q \otimes S) = I_{q+1}$, hence F is surjective.

To show injectivity, assume that x is such that $F(x) = 0$ and $x = \sum_1^q c_i \otimes x_i$, $x_q \neq 0$. Since $f(x_q) \notin N$ and c_q induces an isomorphism between R/N and I_q/I_{q-1}, $c_q f(x_q) \notin I_{q-1}$. On the other hand, F maps the first $q-1$ components into I_{q-1}. Thus $c_q f(x_q) = 0$. Since $c_q f$ is a homogeneous map, we obtain $x_q = 0$.

This completes the proof of Theorem 2.1.

3. Simple Lie conformal superalgebras and their derivations

Here we recall the construction of all simple Lie conformal superalgebras and describe their conformal and ordinary derivations.

3.1. Simple conformal superalgebras. In this subsection we review the main result of [FK].

Example 3.1. Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra. The *loop algebra* associated to $\mathfrak{g}$ is the Lie superalgebra

$$\widetilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}], \;\; p(at^k) = p(a) \;\text{ for }\; a \in \mathfrak{g}, \; k \in \mathbb{Z},$$

with the bracket

$$[a \otimes t^n, b \otimes t^m] = [a, b] \otimes t^{n+m} \quad (a, b \in \mathfrak{g};\; m, n \in \mathbb{Z}).$$

We introduce the family $\mathcal{F}_{\mathfrak{g}}$ of formal distributions (known as currents)

$$a(z) = \sum_{n \in \mathbb{Z}} (a \otimes t^n)\, z^{-n-1}, \quad a \in \mathfrak{g}.$$

It is easily verified that

$$[a(z), b(w)] = [a, b](w)\delta(z - w),$$

hence $(\widetilde{\mathfrak{g}}, \mathcal{F}_{\mathfrak{g}})$ is a formal distribution Lie superalgebra. The associated Lie conformal superalgebra is $\mathbb{C}[\partial] \otimes \mathfrak{g}$, with the λ-bracket (we identify $1 \otimes \mathfrak{g}$ with $\mathfrak{g}$)

$$[a_\lambda b] = [a, b], \quad a, b \in \mathfrak{g}.$$

It is called the *current conformal algebra* associated to $\mathfrak{g}$, and is denoted $\mathrm{Cur}\,\mathfrak{g}$. The Lie conformal superalgebra $\mathrm{Cur}\,\mathfrak{g}$ is simple if and only if $\mathfrak{g}$ is a simple Lie superalgebra.

Example 3.2. We define a conformal linear map $L : \mathrm{Cur}\,\mathfrak{g} \to \mathrm{Cur}\,\mathfrak{g}$ by $L_\lambda g = (\partial + \lambda) g$ for any $g \in \mathfrak{g}$. It is immediate to verify that this is a conformal derivation of $\mathrm{Cur}\,\mathfrak{g}$. L generates the conformal algebra $\mathbb{C}[\partial]L \subset \mathrm{gc}(\mathrm{Cur}\,\mathfrak{g})$ called the *Virasoro conformal algebra* and denoted Vir. The λ-bracket is

$$[L_\lambda L] = (\partial + 2\lambda)L.$$

Thus we have constructed a Lie conformal algebra $\mathrm{Vir} \ltimes \mathrm{Cur}\,\mathfrak{g}$.

Vir can be constructed using formal distributions with coefficients in the Lie algebra $\mathbb{C}[t, t^{-1}]\partial_t$, letting

$$L(z) = \sum_{n \in \mathbb{Z}} (t^n \partial_t) z^{-n-1}.$$

Then,

$$[L(z), L(w)] = \partial_w L(w)\delta(z - w) + 2L(w)\partial_w \delta(z - w).$$

An element of a conformal superalgebra satisfying the above equation is called a *Virasoro element*; it is automatically even.

In all other examples below we forgo the description of conformal superalgebras in terms of formal distributions and simply provide the (conformal) generators and relations. A more detailed description can be found in [FK].

Example 3.3. Recall that for the Grassmann algebra $\wedge(N)$ in the anti-commuting indeterminates ξ_i, $i = 1, \ldots, N$, its Lie superalgebra of derivations is

$$W(N) = \left\{ \sum_{i=1}^{N} P_i \partial_i \mid P_i \in \wedge(N), \partial_i = \partial/\partial\xi_i \right\}.$$

The Lie conformal superalgebra W_N is defined as

$$W_N = \mathbb{C}[\partial] \otimes (W(N) \oplus \wedge(N)).$$

The λ-bracket ($a, b \in W(N); f, g \in \wedge(N)$) is as follows:

$$[a_\lambda b] = [a, b], \quad [a_\lambda f] = a(f) - \lambda(-1)^{p(a)p(f)} fa, \quad [f_\lambda g] = -\partial(fg) - 2\lambda fg. \tag{3.1}$$

The Lie conformal superalgebra W_N is simple for $N \geq 0$ and has rank $(N+1)2^N$. We also remark that $W_0 \simeq \mathrm{Vir}$.

We shall need the following representation of W_N on $\mathbb{C}[\partial] \otimes \wedge(N)$ (cf. Example 3.2):

$$a_\lambda g = a(g), \quad f_\lambda g = -(\partial + \lambda) fg, \quad a \in W(N); f, g \in \wedge(N). \tag{3.2}$$

As a consequence, by identifying $(\mathrm{Cur}\, \mathfrak{g}) \otimes \wedge(N)$ and $\mathfrak{g} \otimes \mathbb{C}[\partial] \otimes \wedge(N)$, we also get a representation of W_N on $\mathrm{Cur}\, \mathfrak{g} \otimes \wedge(N)$ by conformal derivations. Thus, we can construct the semi-direct product $W_N \ltimes (\mathrm{Cur}\, \mathfrak{g} \otimes \wedge(N))$.

Recall that $W(N)$ and $\wedge(N)$ are graded, $W(N) = \oplus_{j \geq -1} W(N)^j$ and $\wedge(N) = \oplus_{j \geq 0} \wedge (N)^j$, with $\deg \xi_i = 1$, $\deg \partial_i = -1$. A subalgebra L of $W(N)$ is said to *act transitively* on $\wedge(N)$ or $\mathfrak{g} \otimes \wedge(N)$ if under the projection $W(N) \to W(N)^{-1}$, L maps onto $W(N)^{-1}$.

Similarly, a subalgebra L of W_N *acts transitively* on $\mathrm{Cur}\, \mathfrak{g} \otimes \wedge(N)$ if the projection of L to $\mathbb{C}[\partial] \otimes W(N)^{-1}$ has rank N.

Example 3.4. For an element $D = \sum_{i=1}^{N} P_i(\partial, \xi)\partial_i + f(\partial, \xi) \in W_N$, we define the corresponding notion of divergence:

$$\mathrm{div}\, D = \sum_{i=1}^{N} (-1)^{p(P_i)} \partial_i P_i - \partial f \in \mathbb{C}[\partial] \otimes \wedge(N).$$

The following identity holds in $\mathbb{C}[\partial] \otimes \wedge(N)$, where $D_1, D_2 \in W_N$ (cf. (3.2)):

$$\mathrm{div}[{D_1}_\lambda D_2] = (D_1)_\lambda (\mathrm{div}\, D_2) - (-1)^{p(D_1)p(D_2)} (D_2)_{-\lambda-\partial} (\mathrm{div}\, D_1). \tag{3.3}$$

Therefore we can define the following subalgebra of W_N:

$$S_N = \{D \in W_N \mid \mathrm{div}\, D = 0\}$$

The Lie conformal superalgebra S_N is simple for $N \geq 2$ and has rank $N 2^N$.

Example 3.5. Let $D = \sum_{i=1}^{N} P_i(\partial, \xi)\partial_i + f(\partial, \xi)$ be an element of W_N. Given $a \in \mathbb{C}$, we define the deformed divergence to be

$$\mathrm{div}_a\, D = \mathrm{div}\, D + af.$$

It still satisfies formula (3.3), hence

$$S_{N,a} = \{D \in W_N \mid \mathrm{div}_a\, D = 0\}$$

is a subalgebra of W_N, which is simple for $N \geq 2$ and has rank $N 2^N$.

Example 3.6. Another variation on the construction of the conformal superalgebra S_N is the following definition (N even):

$$\widetilde{S}_N = \{D \in W_N \mid \operatorname{div}((1+\xi_1 \dots \xi_N)D) = 0\} (= (1-\xi_1 \dots \xi_N) S_N).$$

We thus obtain the Lie conformal superalgebra $\widetilde{S}_N$. It is simple for $N \geq 2$ and has rank $N 2^N$.

Example 3.7. We can also define the "contact" conformal superalgebra

$$K_N = \mathbb{C}[\partial] \otimes \wedge(N)$$

with the λ-bracket for $A = \xi_{i_1} \dots \xi_{i_r}$ and $B = \xi_{j_1} \dots \xi_{j_s}$ defined as

$$[A_\lambda B] = \left(\left(\frac{r}{2} - 1 \right) \partial(AB) + (-1)^r \frac{1}{2} \sum_{i=1}^{N} \partial_i A \partial_i B \right) + \lambda \left(\frac{r+s}{2} - 2 \right) AB.$$

K_N embeds into W_N but we will not be using this fact here.

Note also that $K_0 \simeq \mathrm{Vir}$ and $K_2 \simeq W_1$.

The Lie conformal superalgebra K_N is simple for all $N \in \mathbb{Z}_+$, $N \neq 4$ and is a free $\mathbb{C}[\partial]$-module of rank 2^N.

Example 3.8. The Lie conformal superalgebra K_4 is not simple but its derived subalgebra K_4' is. Furthermore, $K_4 = K_4' \oplus \mathbb{C}\nu$, where $\nu = \xi_1\xi_2\xi_3\xi_4$ and, since K_4' is an ideal in K_4, the map $\operatorname{ad}\nu$ is an outer conformal derivation of K_4'.

Example 3.9. The Lie conformal superalgebra CK_6 is a simple rank 32 subalgebra of K_6, spanned over $\mathbb{C}[\partial]$ by the elements

$$-1 + \alpha\partial^3\nu, \quad \xi_i - \alpha\partial^2\xi_i^*, \quad \xi_i\xi_j - \alpha\partial(\xi_i\xi_j)^*, \quad \xi_i\xi_j\xi_k - \alpha(\xi_i\xi_j\xi_k)^*,$$

where $\alpha \in \mathbb{C}$ is a fixed number such that $\alpha^2 = -1$, $\nu = \xi_1 \dots \xi_6$, and $(\xi_{i_1}\xi_{i_2}\dots)^* = \partial_{i_1}\partial_{i_2}\dots\nu$.

The even part of CK_6 is $\mathrm{Vir} \ltimes \mathrm{Cur}\, \mathfrak{so}_6$ for the Virasoro element $-1+\alpha\partial^3\nu$. For the explicit form of the commutation relations of and further details on CK_6, see [CK].

The main result of [FK] is the following Theorem.

Theorem 3.10. *Any finite simple Lie conformal superalgebra R is isomorphic to one of the Lie conformal superalgebras of the following list:*

(1) W_N, $(N \geq 0)$;
(2) $S_{N,a}$ $(N \geq 2,\ a \in \mathbb{C})$;
(3) $\widetilde{S}_N$ (N *even*, $N \geq 2$);
(4) K_N $(N \geq 0,\ N \neq 4)$;
(5) K_4';
(6) CK_6;
(7) $\mathrm{Cur}\,\mathfrak{s}$, *where* $\mathfrak{s}$ *is a simple finite-dimensional Lie superalgebra.*

We shall call the algebras (1)–(6) from the above list the Lie conformal superalgebras *of Cartan type.*

3.2. **Conformal derivations of simple Lie conformal superalgebras.** We are going describe the conformal derivations of simple Lie conformal superalgebras. With this in mind, for every finite simple Lie conformal superalgebra R we will fix a *distinguished* reductive Lie subalgebra $\mathfrak{r}$ of $(R, {}_{(0)})$.

For $R = \mathrm{Cur}\,\mathfrak{s}$, we choose as $\mathfrak{r}$ the maximal reductive subalgebra of $\mathfrak{s}$ (they are listed in [K2]).

For W_N we take as $\mathfrak{r}$ the copy of $\mathfrak{gl}_N$ spanned by $\xi_i\partial_j \in W(N)$. We construct other distinguished subalgebras in a similar fashion. For S-type superalgebras, we take the subalgebras $\mathfrak{sl}_N$ of $\mathfrak{gl}_N \subset W(N)$ and for the K-type, we take $\mathfrak{r} = \mathfrak{so}_N$ spanned by $\xi_i\xi_j$. For K_4', $\mathfrak{r} = \mathfrak{so}_4 \oplus \mathbb{C}\partial\nu$ and for CK_6, the algebra $\mathfrak{so}_6$ contained in $\mathrm{Cur}\,\mathfrak{so}_6$. Summing up, we obtain the following list of $\mathfrak{r}$'s:

$$\begin{aligned} W_N,\ N\geq 0 \quad &: \quad \mathfrak{gl}_N \\ S_{N,a},\ N\geq 2,\ a\in\mathbb{C} \quad &: \quad \mathfrak{sl}_N \\ \widetilde{S}_N,\ N\geq 2,\ N \text{ even} \quad &: \quad \mathfrak{sl}_N \\ K_N,\ N\geq 0,\ N\neq 4 \quad &: \quad \mathfrak{so}_N \\ K_4' \quad &: \quad \mathfrak{cso}_4 \\ CK_6 \quad &: \quad \mathfrak{so}_6 \end{aligned} \tag{3.4}$$

Proposition 3.11. *In the following we list all cases in which a finite simple Lie conformal superalgebra has outer conformal derivations.*

(1) $\mathrm{Cder}\,K_4' = \mathrm{Cinder}\,K_4' \oplus \mathbb{C}[\partial]\nu$, *where* $\nu = \xi_1\ldots\xi_4 \in \wedge(4)$*;*
(2) $\mathrm{Cder}(\mathrm{Cur}\,\mathfrak{s}) = \mathrm{Vir} \ltimes \mathrm{Cur}(\mathrm{Der}\,\mathfrak{s})$*, where* $\mathfrak{s}$ *is a simple finite-dimensional Lie superalgebra.*

Proof. Let R be a finite simple Lie conformal superalgebra. Then $R = \mathbb{C}[\partial]\otimes U$ and the distinguished reductive Lie algebra $\mathfrak{r}$ acts via the 0-th product completely reducibly on U. This action commutes with ∂, hence $\mathfrak{r}$ acts completely reducibly on R.

Next, $\mathrm{Cder}\,R \subset \mathrm{gc}\,R \simeq R\otimes R^*$ as an R-module, where R^* is the conformal dual of the R-module R [BKL, Proposition 6.4(a)]. Therefore $\mathrm{gc}\,R$ and, hence, $\mathrm{Cder}\,R$ are completely reducible as $\mathfrak{r}$-modules.

Thus, $\mathrm{Cder}\,R = \mathrm{Cinder}\,R \oplus V$ as $\mathfrak{r}$-modules. We will refer to the elements in V as *outer* conformal derivations of R.

Since $[\mathfrak{r}_{(0)}V] \subset R = \mathrm{Cinder}\,R$, we see that $\mathfrak{r}$ kills V, thus outer conformal derivations are $\mathfrak{r}$-module homomorphisms. Using this, outer conformal derivations of R can be easily computed.

In the following we provide a detailed computation in the most involved case, that of W_N. We restrict our attention to $N\geq 2$, the remaining cases being straightforward.

Recall that $W_N = \mathbb{C}[\partial]\otimes(W(N)\oplus\wedge(N))$ and that W_N is completely reducible as a $\mathfrak{gl}_N$-module. We have the following description of $W(N)^k\oplus\wedge(N)^k$ as $\mathfrak{sl}_N(\subset\mathfrak{r})$-modules. Here $R(\lambda)$ denotes the $\mathfrak{sl}_N$-module with the the highest weight λ and π_i are fundamental weights:

$$W(N)^k\oplus\wedge(N)^k = \begin{cases} R(\pi_{N-1}), \quad \text{if } k=-1, \\ R(\pi_{k+1}+\pi_{N-1})\oplus R(\pi_k)\oplus R(\pi_k), \quad \text{if } 0\leq k\leq N-2, \\ R(\pi_{N-1})\oplus R(\pi_{N-1}), \quad \text{if } k=N-1, \\ R(0), \quad \text{if } k=N. \end{cases}$$

Let $\phi \in V$. Then ϕ is a $\mathfrak{gl}_N$-module homomorphism, hence it commutes with the Euler operator $E = \sum \xi_i \partial_i$ and therefore leaves each graded component invariant. The grading being consistent with parity, ϕ can only be an even conformal derivation. Let $x_k = \xi_1 \dots \xi_k E$ be the highest weight vector of $R(\pi_k) \subseteq W(N)^k$, $0 \leq k \leq N-1$. Let $z_k = \xi_1 \dots \xi_{k+1} \partial_N$ be the highest weight vector of $R(\pi_{k+1} + \pi_{N-1}) \subseteq W(N)^k$, $0 \leq k \leq N-2$. Let ∂_N be the highest weight vector of $R(\pi_{N-1}) \subseteq W(N)^{-1}$. Let $y_k = \xi_1 \dots \xi_k$ be the highest weight vector of $R(\pi_k) \subseteq \wedge(N)^k$, $0 \leq k \leq N$. The action of ϕ is given by

$$
\begin{aligned}
&\phi_\lambda y_N = A(\partial, \lambda) y_N, \\
&\phi_\lambda x_k = P_k(\partial, \lambda) x_k + Q_k(\partial, \lambda) y_k, \quad 0 \leq k \leq N-1, \\
&\phi_\lambda y_k = R_k(\partial, \lambda) x_k + S_k(\partial, \lambda) y_k, \quad 0 \leq k \leq N-1, \\
&\phi_\lambda z_k = Z_k(\partial, \lambda) z_k, \quad 0 \leq k \leq N-2, \\
&\phi_\lambda \partial_N = \Omega(\partial, \lambda) \partial_N.
\end{aligned}
$$

Let $g \in \mathfrak{sl}_N$. We compute $\phi_\lambda [g_\mu x_k]$:

$$
\begin{aligned}
\phi_\lambda [g_\mu x_k] =& (\phi_\lambda g)_{\lambda+\mu} x_k + g_\mu (P_k(\partial, \lambda) x_k + Q_k(\partial, \lambda) y_k) \\
=& (\phi_\lambda g)_{\lambda+\mu} x_k + P_k(\partial + \mu, \lambda)(g_\mu x_k) + Q_k(\partial + \mu, \lambda)(g_\mu y_k).
\end{aligned}
$$

Since for any $g \in \mathfrak{sl}_N$, there exists g' such that $g = [g', z_0]$, we have $\phi_\lambda g = \phi_\lambda (g'_{(0)} z_0) = Z_0(\partial, \lambda) g$ as ϕ_λ and $g'_{(0)}$ commute. Then using (3.1) we get

$$
\phi_\lambda [g, x_k] = Z_0(-\lambda - \mu, \lambda)[g, x_k] + P_k(\partial + \mu, \lambda)[g, x_k] + Q_k(\partial + \mu, \lambda)(g y_k - \mu y_k g).
$$

By commuting ϕ_λ and $g_{(0)}$, we get that the left-hand side of the above equality equals $P_k(\partial, \lambda)[g, x_k] + Q_k(\partial, \lambda) g y_k$.

There exists $g \in \mathfrak{sl}_N$ such that $y_k g \neq [g, x_k]$ (e.g. $g = \xi_1 \partial_1$). Hence, $Q_k(\partial + \mu, \lambda) = 0$ implying that $Q_k = 0$. Also, by comparing the left-hand and the right-hand sides, we see that $P_k(\partial, \lambda)$ is at most linear in ∂, otherwise it is impossible to cancel products of μ and ∂ that may come from $P_k(\partial + \mu, \lambda)$.

The rest of the deduction is similar and we describe it only in brief. We compute $\phi_\lambda [x_{0\mu} x_k]$, $\phi_\lambda [y_{0\mu} y_k]$ and $\phi_\lambda [g_\mu y_k]$ to show that $R_k(\partial, \lambda)$ is constant in ∂ whereas $Z_k(\partial, \lambda)$ and $S_k(\partial, \lambda)$ are at most linear in ∂. Third, we compute $\phi_\lambda [x_{0\mu} y_k]$, $\phi_\lambda [x_{0\mu} \partial_N]$, $\phi_\lambda [\partial_{N\mu} y_0]$, $\phi_\lambda [x_{k\mu} y_0]$, $\phi_\lambda [x_{0\mu} y_N]$ and $\phi_\lambda [\partial_{N\mu} \xi_N]$ to show that all the remaining polynomials $A, P_k, R_k, S_k, Z_k, \Omega_k$ are actually zero. □

Remark 3.12. Using the same arguments as in the proof of Proposition 3.11, one can calculate the ordinary derivations of finite simple Lie conformal superalgebras. Below we list all cases in which outer ordinary derivations occur (E denotes the Euler operator and ν, the highest monomial in $\wedge(N)$):

(1) $\operatorname{Der} S_{N,a} = \operatorname{Inder} S_{N,a} \oplus \mathbb{C} E_{(0)} \quad a \neq 0, N \geq 2$;
(2) $\operatorname{Der} S_{N,0} = \operatorname{Inder} S_{N,0} \oplus \mathbb{C} E_{(0)} \oplus \mathbb{C} \nu_{(0)}, \quad N > 2$;
(3) $\operatorname{Der} S_{2,0} = \operatorname{Inder} S_{2,0} \oplus \mathfrak{sl}_2$, where $E_{(0)}, \nu_{(0)} \in \mathfrak{sl}_2$;
(4) $\operatorname{Der} \widetilde{S}_N = \operatorname{Inder} \widetilde{S}_N \oplus \mathbb{C} \nu_{(0)} \quad N \geq 2$, N even;
(5) $\operatorname{Der} K'_4 = \operatorname{Inder} K'_4 \oplus \mathbb{C} \nu_{(0)}$;
(6) $\operatorname{Der} \operatorname{Cur} \mathfrak{s} = \mathbb{C} \partial \oplus \operatorname{Der} \mathfrak{s}$, where $\mathfrak{s}$ is a simple finite-dimensional Lie superalgebra.

We will not require this result for the rest of the paper and so leave the details to the reader.

It follows that the ordinary derivations of simple finite Lie conformal superalgebras form the following Lie superalgebras:

(1) $\operatorname{Der} W_N \simeq W(N) \ltimes \wedge(N)$;
(2) $\operatorname{Der} S_{N,a} \simeq (S(N) \oplus \mathbb{C}E) \ltimes \wedge(N)'$, where $\wedge(N)'$ is the subalgebra of $\wedge(N)$ spanned by monomials of degree strictly less than N, $a \neq 0$;
(3) $\operatorname{Der} S_{N,0} \simeq (S(N) \oplus \mathbb{C}E) \ltimes \wedge(N)$, $N > 2$;
(4) $\operatorname{Der} S_{2,0} \simeq \mathfrak{so}_4 \ltimes \mathcal{H}_4$, where $\mathcal{H}_4$ is the Heisenberg algebra generated by $\xi_1, \xi_2, \partial_1, \partial_2$, and $\mathfrak{so}_4$ is spanned by two $\mathfrak{sl}_2$-triples: $(\xi_2\partial_1, \xi_1\partial_1 - \xi_2\partial_2, \xi_1\partial_2)$, and (ν, E, F), where $F(\xi_1) = -\partial_2$, $F(\xi_2) = \partial_1$.
(5) $\operatorname{Der} \widetilde{S}_N \simeq ((1-\nu)S(N)) \ltimes \wedge(N)$, N even;
(6) $\operatorname{Der} K_N$, $N \neq 4$, is the natural central extension of $H(N)$ by a one-dimensional center;
(7) $\operatorname{Der} K_4' \simeq H(4)$;
(8) $\operatorname{Der} CK_6$ is the central extension of the "strange" Lie superalgebra $P(4)$ by a one-dimensional center;
(9) $\operatorname{Der} \operatorname{Cur} \mathfrak{s}$ is the direct sum of $\operatorname{Der} \mathfrak{s}$ with the one-dimensional Lie algebra.

(Cf. the calculations of derivations of corresponding formal distribution Lie superalgebras in [K4] and [FK].)

4. Conformal derivations of differentiably simple Lie conformal superalgebras

Here we describe conformal derivations of Lie conformal superalgebras of the form $S \otimes \wedge(n)$, where S is a finite simple Lie conformal superalgebra. By Theorem 2.1 this will take care of the conformal derivations of all differentiably simple Lie conformal superalgebras.

4.1. **Conformal centroid.** The *conformal centroid* of a Lie conformal superalgebra R is the subalgebra $CC(R)$ of the associative conformal algebra $\operatorname{Cend} R$ defined by

$$CC(R) = \{\varphi \in \operatorname{Cend} R \mid \varphi_\lambda[x_\mu y] = [(\varphi_\lambda x)_{\lambda+\mu} y]\}.$$

Remark 4.1. A direct calculation similar to that in the proof of Lemma 2.6 shows that for $\varphi \in CC(R)$, $\varphi_\lambda[x_\mu y] = (-1)^{p(x)p(\varphi)}[x_\mu(\varphi_\lambda y)]$.

To describe conformal centroids of simple superalgebras, we need two technical lemmas:

Lemma 4.2. *Let R be a finite Lie conformal superalgebra. Suppose R contains a Virasoro element L and a $\mathbb{C}[\partial]$-basis $\{a_j\}_{j\in J}$ of R such that*

$$[L_\lambda a_j] = (\partial + \Delta_j\lambda)a_j + b_j \quad \textit{for any} \quad j \in J, \tag{4.1}$$

where $\Delta_j \in \mathbb{C}$, $b_j \in R$ and, whenever $\Delta_j = 0$, $b_j = 0$.
Then $CC(R) = 0$.

Proof. Let $\varphi \in CC(R)$. Suppose $\deg_\lambda \varphi_\lambda L = k$. Then $\deg_\lambda[L_\mu(\varphi_\lambda L)] = k$ and $\deg_\lambda \varphi_\lambda[L_\mu L] = \deg_\lambda(\partial + \lambda + 2\mu)\varphi_\lambda L = k+1$ unless $\varphi_\lambda L = 0$. Hence the equation $\varphi_\lambda[L_\mu L] = [L_\mu(\varphi_\lambda L)]$, given in Remark 4.1 implies that $\varphi_\lambda L = 0$.

Now, for any $j \in J$, we have

$$\varphi_\lambda[L_\mu a_j] = \varphi_\lambda((\partial + \Delta_j\mu)a_j + d(a_j)).$$

On the other hand,

$$\varphi_\lambda[L_\mu a_j] = [(\varphi_\lambda L)_{\lambda+\mu} a_j] = 0.$$

Therefore,

$$(\partial + \lambda + \Delta_j \mu)\varphi_\lambda a_j + \varphi_\lambda b_j = 0.$$

Now, if $\Delta_j \neq 0$ the fact that there is no power of μ in the second summand implies that $\varphi_\lambda a_j = 0$. If $\Delta_j = 0$, then, by assumption, $b_j = 0$, hence $\varphi_\lambda a_j = 0$. □

Now we present a method of constructing a basis satisfying requirements of Lemma 4.2 which we will apply in the case of simple Lie conformal superalgebras of Cartan type.

Lemma 4.3. *Let R be a finite simple Lie conformal superalgebra, $R = \mathbb{C}[\partial] \otimes V$. Let $\mathfrak{r}$ be the distinguished reductive Lie subalgebra of R and let L be a Virasoro element of R such that*

$$[L_\lambda g] = (\partial + \lambda) g \quad \text{for all } g \in \mathfrak{r}. \tag{4.2}$$

Suppose $V = \oplus_{i \in I} V_i$, where V_i is an irreducible $\mathfrak{r}$-module whose highest weight vector is v_i. Suppose that the v_i satisfy (4.1). Then there exists a $\mathbb{C}[\partial]$-basis $\{a_j\}_{j \in J}$ of R satisfying conditions of Lemma 4.2.

Proof. For any $i \in I$, V_i is spanned over $\mathbb{C}$ by vectors of the form $v_i^n = g^n_{(0)} \cdots g^1_{(0)} v_i$, $n \in \mathbb{Z}_+$ and $g^k \in \mathfrak{r}$ for any $k = 1, \ldots, n$.

We are going to prove by induction on n that

$$[L_\lambda v_i^n] = (\partial + \Delta_i \lambda) v_i^n + u_i^n,$$

where $u_i^n \in R$, and whenever $\Delta_i = 0$, $u_i^n = 0$ for all n. Then it would suffice to take v_i^n's as the necessary basis. We have

$$\begin{aligned}
[L_\lambda (g^{n+1}_{(0)} v_i^n)] &= [[L_\lambda g^{n+1}]_\lambda v_i^n] + [g^{n+1}_{(0)} [L_\lambda v_i^n]] \\
&= [((\partial + \lambda) g^{n+1})_\lambda v_i^n] + [g^{n+1}_{(0)} ((\partial + \Delta_i \lambda) v_i^n + u_i^n] \\
&= (\partial + \Delta_i \lambda)[g^{n+1}_{(0)} v_i^n] + [g^{n+1}_{(0)} u_i^n].
\end{aligned}$$

Then we take $u_i^{n+1} = [g^{n+1}_{(0)} u_i^n]$. This completes the proof. □

Proposition 4.4. *Let R be a finite simple Lie conformal superalgebra.*

(1) *If R is of Cartan type, then $CC(R) = 0$;*
(2) *If $R = \mathrm{Cur}\,\mathfrak{s}$, then $CC(R) = \{\varphi \in \mathrm{Cend}\, R \mid \varphi_\lambda s = p(\partial, \lambda) s, p(\partial, \lambda) \in \mathbb{C}[\partial, \lambda], \text{for all } s \in \mathfrak{s}\}$.*

Proof. If R is of Cartan type, then it has an element L and a $\mathbb{C}[\partial]$-basis satisfying (4.1) and (4.2) (such an L is called physical in Section 6). In fact, if R is not $S_{N,a}$ or $\widetilde{S}_N$, we can choose $L = -1$ (and then all $b_j = 0$). In the case of $S_{N,a}$ or $\widetilde{S}_N$, we choose $L = -1 + \frac{1}{N}(\partial - a)E$ and $-1 + \xi_1 \ldots \xi_N + \frac{1}{N}\partial E$, respectively. The formulas that appear in the proof of [FK, Proposition 4.16] show that the highest weight vectors satisfy (4.1) with Δ_i taking the values of $2/N, 3/N, \ldots, (N-1)/N, 1, (N+1)/N, \ldots, (2N-1)/N, 2$. Consequently, by Lemma 4.3 we can find a $\mathbb{C}[\partial]$-basis whose conformal weights are nonzero. We complete the proof of (1) by applying Lemma 4.2.

Let $\varphi \in CC(\mathrm{Cur}\,\mathfrak{s})$. Then for any $s, s' \in \mathfrak{s}$ we have

$$\varphi_\lambda [s_{(0)} s'] = (-1)^{p(\varphi)p(s)} [s_{(0)} \varphi_\lambda s'].$$

Suppose $\varphi_\lambda = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}\varphi_{(n)}$. Then for any $n \geq 0$, $\varphi_{(n)}$ is a $\mathfrak{s}$-module homomorphism. Since $\mathfrak{s}$ is simple, if $\varphi_{(n)} \neq 0$, then $\ker\varphi_{(n)} = 0$ and $\operatorname{im}\varphi_{(n)} \simeq \mathfrak{s}$. Then $\phi_{(n)}s = \sum_i p_{n_i}(\partial)s_i(s)$. The map $s \mapsto s_i(s)$ is an $\mathfrak{s}$-automorphism. Remark 2.10 implies that $s_i(s) = c_i s$ for some $c_i \in \mathbb{C}$. Hence, $\varphi_{(n)}(s) = p_n(\partial)\otimes s$ for any $s \in \mathfrak{s}$ and $\varphi_\lambda = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!} p_n(\partial)\otimes 1_{\mathfrak{s}}$. □

Remark 4.5. In fact, we have shown that $CC(\operatorname{Cur}\mathfrak{s}) \simeq \operatorname{Cend}_1$. We note that this conformal algebra is neither finite, nor commutative.

4.2. **Conformal derivations of a tensor product.**

Proposition 4.6. *Let R be a finite Lie conformal superalgebra and let B be a unital commutative associative finite-dimensional superalgebra. Then*

(1) $\operatorname{Cder} R \otimes B \subseteq \operatorname{Cder}(R\otimes B)$.
(2) $CC(R)\otimes \operatorname{Der} B \subseteq \operatorname{Cder}(R\otimes B)$.
(3) *If R is simple and of Cartan type, then*
$$\operatorname{Cder}(R\otimes B) = \operatorname{Cder} R\otimes B.$$
(4) *If $R = \operatorname{Cur}\mathfrak{s}$, where $\mathfrak{s}$ is a simple finite-dimensional Lie superalgebra, then*
$$\operatorname{Cder}(\operatorname{Cur}\mathfrak{s}\otimes B) = \operatorname{Cder}(\operatorname{Cur}\mathfrak{s})\otimes B + 1_{\mathfrak{s}}\otimes \operatorname{Cder}\operatorname{Cur} B.$$
In particular,
$$\operatorname{Cder}(\operatorname{Cur}\mathfrak{s}\otimes \wedge(N)) = \operatorname{Cur}(\operatorname{Der}\mathfrak{s})\otimes\wedge(N) + 1_{\mathfrak{s}}\otimes W_N.$$

Proof. (1) Let $\phi \in \operatorname{Cder} R$ and $b\in B$. We set
$$(\phi\otimes b)_\lambda(r'\otimes b') = (-1)^{p(b)p(r')}(\phi_\lambda r')\otimes bb'.$$
It is easy to verify that $\phi\otimes b$ is a conformal derivation.

(2) Similarly, let $\varphi\in CC(R)$ and $d\in\operatorname{Der} B$. We set
$$(\varphi\otimes d)_\lambda(r\otimes b) = (-1)^{p(d)p(r)}(\varphi_\lambda r)\otimes d(b).$$
It is easy to verify that $\varphi\otimes d$ is a conformal derivation.

(3) Let $\phi\in\operatorname{Cder}(R\otimes B)$. For any $r\in R$ we have
$$\phi_\lambda(r\otimes 1) = \sum_{i\in I} \phi_{i\,\lambda}(r)\otimes b_i$$
where $\{b_i\}_{i\in I}$ is a linear basis of B. A short direct computation shows that $\phi_i \in \operatorname{Cder} R$ and $\widetilde{\phi} := \phi - \sum_{i\in I}\phi_i\otimes b_i \in \operatorname{Cder}(R\otimes B)$ is such that $\widetilde{\phi}_\lambda(r\otimes 1) = 0$. Let us fix $b\in B$. Suppose $\widetilde{\phi}_\lambda(r\otimes b) = \sum_{i\in I}\varphi_{i\,\lambda} r\otimes b_i$. For any $r, r'\in R$ we have $[r_\mu r']\otimes b = [(r\otimes 1)_\mu(r'\otimes b)]$. If we apply $\widetilde{\phi}$ to both sides of this equation we see that $\varphi_i\in CC(R)$ which is zero by Proposition 4.4.

(4) Using the same argument as in (3), one can see that $\varphi_i\in CC(\operatorname{Cur}\mathfrak{s})$, so that by Proposition 4.4
$$\widetilde{\phi}_\lambda(r\otimes b) = \sum_{i\in I} P_i(\partial,\lambda) r\otimes b_i.$$
Now, we identify $\operatorname{Cur}\mathfrak{s}\otimes B$ with $\mathfrak{s}\otimes\operatorname{Cur} B$ so that
$$\widetilde{\phi}_\lambda(r\otimes b) = r\otimes\sum_{i\in I} P_i(\partial,\lambda)\otimes b_i.$$
The map associating b to $\sum_{i\in I} P_i(\partial,\lambda)\otimes b_i$ is easily shown to be a conformal derivation of $\operatorname{Cur} B$.

In the case $B = \wedge(N)$, it remains to show that $W_N = \mathrm{Cder}(\mathrm{Cur}\wedge(N))$. We recall at first that W_N acts on $\mathrm{Cur}\wedge(N)$ by conformal derivations (see (3.2)). The rest is done in two steps. First, notice that if $\phi \in \mathrm{Cder}(\mathrm{Cur}\wedge(N))$, then $\phi_\lambda 1 = (\partial+\lambda)\sum_{i\in I} p_i(\lambda) f_i$, where $\{f_i\}_{i\in I}$ is a linear basis of $\wedge(N)$. Second, if we set $\widetilde{\phi} = \phi - (\sum_{i\in I} p_i(-\partial) f_i)\circ\partial$, it is immediate to see that for any $j = 1,\dots,N$, $\widetilde{\phi}_\lambda(\xi_j) = P_j(\lambda,\xi)$, so that $\widetilde{\phi} = \sum_{j=1}^N P_j(-\partial,\xi)\partial_j$. Therefore $\phi = \widetilde{\phi} + (\sum_{i\in I} p_i(-\partial) f_i)\circ\partial \in W_N$. □

We thus obtain

Theorem 4.7. *The following is a complete list of finite non-abelian differentiably simple Lie conformal superalgebras:*

(1) $(\mathrm{Cur}\,\mathfrak{s})\otimes\wedge(n)$, *where* $\mathfrak{s}$ *is a simple finite-dimensional Lie superalgebra;*
(2) *a finite simple Lie conformal superalgebra of Cartan type.*

Proof. If R is differentiably simple, then $R \simeq S\otimes\wedge(n)$ by Theorem 2.1. But if S is of Cartan type, an ideal $S\otimes I$ of R, where I is an ideal of $\wedge(n)$, is differentiably stable by Proposition 4.6(3). Part (4) of the same proposition shows that the conformal superalgebra $(\mathrm{Cur}\,\mathfrak{s})\otimes\wedge(n)$, $\mathfrak{s}$ as above, is differentiably simple. □

5. Semisimple Lie conformal superalgebras

Here we provide a detailed description of finite semisimple Lie conformal superalgebras. In particular, we prove the following

Theorem 5.1. *Let R be a finite semisimple Lie conformal superalgebra. Then R splits into a direct sum of conformal algebras of the following types:*

(1) *a finite simple Lie conformal superalgebra;*
(2) *a Lie conformal superalgebra L such that*

$$\left(\bigoplus_{i=1}^{k}(\mathrm{Cur}\,\mathfrak{s}_i)\otimes\wedge(n_i)\right)\oplus {K_4'}^{\oplus r}\subset L \subset \left(\bigoplus_{i=1}^{k}\left(\mathrm{Cur}(\mathrm{Der}\,\mathfrak{s}_i)\otimes\wedge(n_i)+1_{\mathfrak{s}}\otimes W_{n_i}\right)\right)\oplus {K_4}^{\oplus r}, \tag{5.1}$$

where $n_i, r\in\mathbb{Z}_+$ and $\mathfrak{s}_i$ are simple finite-dimensional Lie superalgebras, and such that for each i the projection of L onto W_{n_i} acts transitively on $(\mathrm{Cur}\,\mathfrak{s}_i)\otimes\wedge(n_i)$.

5.1. **Proof of Theorem 5.1.** Define the *socle* of a Lie conformal algebras as the sum of all its minimal ideals.

Lemma 5.2. *Let R be a finite semisimple Lie conformal superalgebra. Then its socle is a direct sum of all minimal ideals of R and there exist finitely many such ideals in R.*

Proof. Let $\{M_i\}_{i\in I}$ be the family of all minimal ideal of R. By Lemma 1.4 this family is non-empty.

Notice that for all $i,j\in I$, $[M_{i\,\lambda} M_j]=0$ for $i\neq j$ by minimality. Suppose $M_i\cap\sum_{j\neq i}M_j\neq 0$ for some i. Thus by minimality of M_i, $M_i\subseteq\sum_{j\neq i}M_j$, and $[M_{i\,\lambda}M_i]\subset\sum_{j\neq i}[M_{i\,\lambda}M_j]=0$. Hence M_i is abelian which contradicts semisimplicity of R.

Thus the sum of minimal ideals is direct. Furthermore, the rank of a direct sum is the sum of the ranks, hence the fact that R is finite implies that R contains only finitely many minimal ideals. □

We fix notations from the proof and denote the socle of R by M and the minimal ideals of R by M_i, $i = 0, 1, \ldots, l$.

The centralizer of M in R, $C_R(M) = \{x \in R \mid [x_\lambda M] = 0\}$, is zero. This follows from the fact that $C_R(M)$ is an ideal of R, but contains no minimal ideals of R, because any such ideal would be abelian contradicting the semisimplicity of R. But this contradicts Lemma 1.4.

Therefore, the homomorphism of Lie conformal superalgebras $R \to \operatorname{Cder} M$ sending x to $\operatorname{ad}_M x_\lambda$ is injective and we have

$$\operatorname{Cinder} M \subseteq R \subseteq \operatorname{Cder} M.$$

The minimality of M_i implies that $\langle [M_{i\lambda} M_i] \rangle = M_i$ for any $i = 0, 1, \ldots, l$. Thus if $\phi \in \operatorname{Cder} M$ maps M_i to M_j, $\phi(M_i)$ must be zero. It follows that $\operatorname{Cder} M = \oplus_{i=0}^{l} \operatorname{Cder} M_i$.

Therefore, we have

$$\bigoplus_{i=0}^{l} \operatorname{Cinder} M_i \subseteq R \subseteq \bigoplus_{i=0}^{l} \operatorname{Cder} M_i. \tag{5.2}$$

Put $R_i = R \cap \operatorname{Cder} M_i$.

Lemma 5.3. *Let R be a conformal superalgebra satisfying (5.2). Then R is semisimple if and only if M_i is R_i-simple for all i.*

Proof. If M_i is not R_i-simple for some i, then it contains a non-trivial ideal J. By differential simplicity of M_i, J is nilpotent (see Proposition 2.14). Since J is also an ideal of R_i and hence of R, R is not semisimple.

Conversely, let R contain an abelian ideal I. Since M_i is non-abelian and R-simple for any i, $[I_\lambda M_i]$, as an R-submodule of M_i must be zero. Thus I kills M, a contradiction. □

By Lemma 1.5(2), M_i's are differentiably simple and, since they are non-abelian, we obtain a complete description of the possible form of M_i's from Theorem 4.7.

Now we can use Propositions 3.11 and 4.6 to describe possible R_i's in greater detail.

Let $M_i = S_i \otimes \wedge(n_i)$, where S_i is a simple Lie conformal superalgebra.

By Proposition 4.7, if S_i is of Cartan type, $n_i = 0$.

If S_i is of Cartan type and not isomorphic to K_4', then $R_i \simeq S_i$.

If $S_i \simeq K_4'$, $R_i \subset K_4$.

It remains to treat the case of a current Lie conformal algebra S_i. Let R_i be a semisimple Lie conformal superalgebra such that $\operatorname{Cur} \mathfrak{s} \otimes \wedge(n) \subset R_i \subset W_n \ltimes (\operatorname{Cur} \operatorname{Der} \mathfrak{s} \otimes \wedge(n))$, where $\mathfrak{s}$ is a finite-dimensional simple Lie superalgebra. In order to prove that the projection of R_i onto W_{n_i} acts transitively if and only if $S_i \otimes \wedge(n_i)$ is R_i-simple, we need an auxiliary lemma

Recall that a commutative associate superalgebra with the action of a Lie superalgebra $\mathfrak{a}$ by derivations is called $\mathfrak{a}$-differentiably simple if it contains no non-trivial invariant ideals.

Lemma 5.4. *$\wedge(n)$ is $\mathfrak{a}$-simple for a subalgebra $\mathfrak{a}$ of $W(n)$ if and only if $\mathfrak{a}$ acts transitively on $\wedge(n)$.*

Proof. Transitivity of the action of $\mathfrak{a}$ is equivalent to saying that $\mathfrak{a}$ contains elements $a_j = \partial_j + \sum_i p_i \partial_i$, where $p_i \in \wedge(n)$, $p_i(0) = 0$, for all j.

Then, assume that $\mathfrak{a}$ acts transitively and let I be a non-zero $\mathfrak{a}$-stable ideal of $\wedge(n)$. Then I contains the monomial $\xi_1 \dots \xi_n$. Let $f = \xi_{k_1} \dots \xi_{k_n}$. Since $a_j(\xi_j f) = f+$ monomials of higher degree, by induction I contains all monomials of $\wedge(n)$, i.e. $I = \wedge(n)$.

Now let $\wedge(N)$ be $\mathfrak{a}$-differentiably simple. We can assume that $\mathfrak{a}$ is closed with respect to multiplication by $\wedge(N)$. Indeed, $[a, fb] = a(f)b + (-1)^{p(a)+p(f)}[a, b]$ for $a, b \in W(N)$, a homogeneous, so $\wedge(N)\mathfrak{a}$ remains closed with respect to the Lie bracket. Also, $\mathfrak{a}$ acts transitively if and only if $\wedge(N)\mathfrak{a}$ does. Similarly, $\wedge(N)$ is $\mathfrak{a}$-differentiably simple if and only of it is $\wedge(N)$-differentiably simple.

If the projection of $\mathfrak{a}$ to $W(N)^{-1}$ is zero, then by the grading argument, $\mathfrak{a}$ leaves invariant the maximal ideal of $\wedge(N)$. Thus, possibly after a linear transformation of ξ_i's, $\mathfrak{a}$ contains an element of the form $a_N = \partial_N + \sum p_i \partial_i$, $p_i(0) = 0$.

In particular, $a_N = \partial_N + \xi_1 b + \sum q_i \partial_i$, where $b \in W(N)$, $q_i(0) = 0$ and $\partial_1 q_i = 0$. Also, we can assume that $q_N = 0$. We introduce the following change of coordinates: $\xi_i' = \xi_i - \xi_N q_i$ for $1 \le i \le N-1$, $\xi_N' = \xi_N$. Notice that $\mathbb{C}\langle \xi_1', \xi_2 \dots, \xi_N \rangle = \wedge(N)$. Indeed, let $q_1 = \xi_1 f + g$, $\partial_1 g = 0$. Then $\xi_1 = \xi_1(1 + \xi_N f) + \xi_N g$, and since $1 - f$ is invertible, this change of coordinates is valid. By induction, we obtain that $\mathbb{C}\langle \xi_1', \dots, \xi_N' \rangle$. The chain rule $\partial_i = \sum (\partial_i \xi_j')\partial_j'$ implies that in the new coordinates $a_N = \partial_N' + \xi_N c$ for some $c \in W(N)$.

Thus, we can assume that $a_N = \partial_N + \xi_N c$. Then $[a_N, a_N] = 2c$, hence $\partial_N \in \mathfrak{a}$ and we can split $\mathfrak{a}$ as $\wedge(N)\partial_N \oplus \mathfrak{a}_1$. If $\mathfrak{a}_1$ contains an element $a = \xi_N b$, then $[\partial_N, a] = b \in \mathfrak{a}_1$, i.e. we can both multiply by and cancel ξ_N in elements of $\mathfrak{a}_1$. Thus $\mathfrak{a}_1$ splits as $\mathfrak{a}_2 \oplus \xi_N \mathfrak{a}_2$, where $\mathfrak{a}_2 \subset W(N-1)$. If I is a proper $\mathfrak{a}_2$-stable ideal of $\wedge(N-1)$, then clearly $I + \xi_1 I$ is a proper $\mathfrak{a}$-stable ideal. Hence, $\wedge(N-1)$ is $\mathfrak{a}_2$-differentiably simple. Moreover, $\mathfrak{a}$ acts transitively if and only if $\mathfrak{a}_2$ does.

Induction on N completes the proof. $\square$

Remark 5.5. Along the way we proved an analog of the Frobenius theorem for $\wedge(N)$. Namely, if I is an ideal of $\wedge(N)$ and $\mathfrak{a}$ is a subalgebra of $W(N)$ which leaves I invariant, then after a change of coordinates, $\mathfrak{a} = \sum_{i=1}^{k} \wedge(N)\partial_i + \sum_{j=k+1}^{N} \wedge_k(N)\partial_j$, where $\wedge_k(N)$ is the subalgebra (without 1) of $\wedge(N)$ generated by $\xi_{k+1}, \dots, \xi_N$.

The explicit description of the action of W_n on $\mathrm{Cur}\,\mathfrak{s} \otimes \wedge(n)$ in the proof of Proposition 4.6(4) shows that the action of R_i is transitive if and only if the action of $R_i|_{\partial=1}$ is transitive on $\wedge(n)$. This together with Lemmas 5.3 and 5.4 completes the proof of Theorem 5.1.

5.2. **Conformal derivations.** The following is a direct analog of [C, Proposition 7.4]:

Proposition 5.6. *Let R be a finite semisimple Lie conformal superalgebra and let M be its socle. Then $\mathrm{Cder}\, R = \{\phi \in \mathrm{Cder}\, M \mid \langle \phi_\lambda R \rangle \subset R\}$, the normalizer of R in $\mathrm{Cder}\, M$.*

Proof. If I is a minimal $\mathrm{Cder}\, R$-stable ideal, it is either minimal or properly contains an ideal J of R. Since I is differentiably simple, J must be nilpotent, a contradiction. Hence $I \subset M$. Conversely, let M' be the sum of all minimal $\mathrm{Cder}\, R$-stable

ideal and I a minimal ideal such that $I \not\subset M'$. Then $I \subset \operatorname{Ann} M'$ and, as $\operatorname{Ann} M'$ is an ideal of $\operatorname{Cder} R$ and contains no $\operatorname{Cder} R$-central elements, it must contain a minimal $\operatorname{Cder} R$-ideal which must be abelian. The contradiction shows that $M = M'$ and M is $\operatorname{Cder} R$-stable.

Then $\operatorname{Cder} R$ embeds into $\operatorname{Cder} M$ and the rest follows easily. $\square$

6. Physical Virasoro pairs

In this section we compute all physical Virasoro pairs in a finite simple Lie conformal superalgebra. As a consequence, we obtain the classification of physical Lie conformal superalgebras.

6.1. **Definitions.** Let $R = \mathbb{C}[\partial] \otimes V$ be a finite simple Lie conformal superalgebra. Recall that V contains a distinguished reductive subalgebra of V; we denote it $\mathfrak{r}$.

Let R contain a Virasoro element L such that

$$[L_\lambda g] = (\partial + \lambda) g \text{ for any } g \in \mathfrak{r}. \tag{6.1}$$

Then (R, L) is a *simple physical Virasoro pair.*

By skew-symmetry, (6.1) is equivalent to $[g_\lambda L] = \lambda g$, which implies that $[g_{(0)} L] = 0$, i.e. that L is invariant with respect to the action of $\mathfrak{r}$ on R (by means of the 0th product). Also, R is completely reducible as an $\mathfrak{r}$-module because the 0th product commutes with the $\mathbb{C}[\partial]$-module structure.

6.2. **Classification results.**

Theorem 6.1. *The following is a complete list of all simple physical Virasoro pairs:*

$$\begin{array}{rcl}
W_0 & : & L = -1 \\
W_N,\ N \geq 1 & : & L = -1 + (p_0 + p_1 \partial) E \\
S_{N,a},\ N \geq 2,\ a \in \mathbb{C} & : & L = -1 + \dfrac{1}{N}(\partial - a) E \\
\widetilde{S}_N,\ N \geq 2,\ N\, even & : & L = -(1 - \xi_1 \dots \xi_N) + \dfrac{1}{N} \partial E \\
K_N,\ N \geq 1,\ N\, odd & : & L = -1 \\
K_N,\ N > 6,\ N\, even & : & L = -1 + p_1 \partial \nu \\
K_6 & : & L = -1 + (p_1 \partial + p_3 \partial^3) \nu \\
K_4' & : & L = -1 + (p_0 + p_1 \partial) \partial \nu \\
CK_6 & : & L = -1 + \alpha \partial^3 \nu .
\end{array}$$

where $E = \sum_{i=1}^N \xi_i \partial_i$ is the Euler operator, $\nu = \xi_1 \dots \xi_N$ is the longest monomial in $\wedge(N)$, $p_0, p_1, p_3 \in \mathbb{C}$ and $\alpha \in \mathbb{C}$, $\alpha^2 = -1$.

Proof. Recall that $W_N = \mathbb{C}[\partial] \otimes (W(N) \oplus \wedge(N))$. If $N = 0$, the equation $[L_\lambda L] = (\partial + 2\lambda) L$ implies that $L = -1$. Suppose $N \geq 1$, then $\mathfrak{r} = \mathfrak{gl}_N$. First of all we remark that L cannot be contained in the radical of the even part of W_N, which we denote by $\operatorname{Rad} W_{N\,\bar{0}}$. We set

$$S = \mathbb{C}[\partial] \otimes (\oplus_{k \geq 2,\ k \text{ even}} W(N)^k), \quad T = \mathbb{C}[\partial] \otimes (\oplus_{k \geq 2,\ k \text{ even}} \wedge^k (N)).$$

Since the λ-bracket in W_N is graded, $S \oplus T \subseteq \operatorname{Rad} W_{N\ \bar{0}}$, hence any L must have a component in $\mathbb{C}[\partial] \otimes (W(N)^0 \oplus \mathbb{C}1)$:

$$L = Q(\partial)1 + P(\partial)E + \sum_i P_i(\partial)g_i + s + t,$$

where $\{g_i\}$ is a basis of $\mathfrak{sl}_N$, $s \in S$ and $t \in T$. Furthermore, we can compare the coefficients of the degree 0 basis elements in the equation $[L_\lambda L] = (\partial + 2\lambda)L$. In the case of $1 \in \wedge(N)$ we have:

$$-(\partial + 2\lambda)Q(-\lambda)Q(\partial + \lambda) = (\partial + 2\lambda)Q(\partial),$$

hence $Q(\partial) = Q$ is constant and either $Q = 0$ or $Q = -1$. Comparing the coefficients of E we get:

$$-(\partial + \lambda)QP(\partial + \lambda) - \lambda QP(-\lambda) = (\partial + 2\lambda)P(\partial).$$

Comparing the terms with basis elements in $\mathfrak{sl}_N$ we get

$$(\partial + \lambda)Q \sum_i P_i(\partial + \lambda)g_i - \lambda Q \sum_i P_i(-\lambda)g_i + \sum_{i,j} P_i(-\lambda)P_j(\partial + \lambda)[g_i, g_j]$$
$$= (\partial + 2\lambda) \sum_i P_i(\partial)g_i.$$

Now, $Q = 0$ implies $P = 0$ and

$$\sum_{i,j} P_i(-\lambda)P_j(\partial + \lambda)[g_i, g_j] = (\partial + 2\lambda) \sum_i P_i(\partial)g_i.$$

If we examine the degree of ∂ in the last equation, we see that $P_i = 0$ for any i. It follows that if $Q = 0$, then $L = s + t \in \operatorname{Rad} W_{N\ \bar{0}}$, which is impossible. Hence $Q = -1$. If we examine the equation for E, which is homogeneous, we conclude that $P(\partial) = p_0 + p_1\partial$ and $L = -1 + (p_0 + p_1\partial)E + \sum_i P_i(\partial)g_i + s + t$.

Next, we impose the condition that $[L_\lambda g] = (\partial + \lambda)g$ for any $g \in \mathfrak{gl}_N$. Since $[-1_\lambda g] = (\partial + \lambda)g$, we have $[(\sum_i P_i(\partial)g_i + s + t)_\lambda g] = 0$ for any $g \in \mathfrak{gl}_N$. Consequently, in degree 0 we have $[(\sum_i P_i(\partial)g_i)_\lambda g] = 0$ for any $g \in \mathfrak{gl}_N$. In particular, it follows that $\sum_i P_i(\partial)g_i$ lies in the center of $\operatorname{Cur}\mathfrak{sl}_N$ which is zero. On the other hand, in degree ≥ 2 we have $[(s + t)_\lambda g] = 0$, hence $[g_{(0)}(s + t)] = 0$. By looking at the decomposition of the $\mathbb{C}[\partial]$-basis of W_N into irreducible $\mathfrak{gl}_N$-modules, we see that there are no invariants in degree ≥ 2. Consequently, $s + t = 0$ and we conclude that $L = -1 + (p_0 + p_1\partial)E$.

A similar argument, with $\mathfrak{sl}_N$ replacing $\mathfrak{gl}_N$, provides the solution for $S_{N,a}$ and $\widetilde{S}_N$.

The cases of K_N, $0 \leq N \leq 3$, K_4' and CK_6 are dealt with in a similar fashion. As for K_N, $N \geq 5$, the unique $\mathfrak{so}_N$-invariant is $\wedge^N(N)$. If N is odd, this vector is odd too, hence cannot appear in L. If N is even, from $[L_\lambda L] = (\partial + 2\lambda)L$ we obtain a homogeneous equation, which admits a linear solution for any N and a cubic solution for $N = 6$.

In the case of $\operatorname{Cur}\mathfrak{s}$, there are no non-zero Virasoro elements. To see this, let $\{g_i\}_{i\in I}$ be a linear basis of $\mathfrak{s}$. Then $L = \sum_i P_i(\partial)g_i$ and by substituting this expression into $[L_\lambda L] = (\partial + 2\lambda)L$ and comparing the degrees in ∂, we see that $L = 0$. □

Corollary 6.2. *The following is a complete list of all pairs (R, L), where R is a finite simple Lie conformal superalgebra and L is a Virasoro element such that*

$R_{\bar{0}} = L \ltimes \operatorname{Cur} \mathfrak{g}$, *where* $\mathfrak{g}$ *is a finite-dimensional Lie algebra and* $[L_\lambda a] = (\partial + \lambda)a$ *for all* $a \in \mathfrak{g}$*:*

$$\begin{aligned}
W_1, &\quad -1 + (p_0 + p_1\partial)E\\
W_2, &\quad -1 + \frac{1}{2}\partial E\\
S_{2,a}, &\quad -1 + \frac{1}{2}(\partial - a)E\\
\widetilde{S}_2, &\quad -(1-\nu) + \frac{1}{2}\partial E\\
K_0, &\quad -1\\
K_1, &\quad -1\\
K_3, &\quad -1\\
K_4', &\quad -1 + (p_0 + p_1\partial)\partial\nu\\
CK_6, &\quad -1 + \alpha\partial^3\nu.
\end{aligned}$$

Proof. We have to impose the condition $[L_\lambda g] = (\partial + \lambda)g$ for any $g \in \mathfrak{g}$.

Let $w \in W(0,N)_k \subseteq W_N$ and $f \in \wedge^k(N) \subseteq W_N$. We have

$$\begin{aligned}
[L_\lambda w] &= (p_0 k + \partial + (1 - p_1 k)\lambda)w,\\
[L_\lambda f] &= (p_0 k + \partial + (2 - p_1 k)\lambda)f - \lambda(p_0 - p_1\lambda)fE.
\end{aligned}$$

If $N > 2$, $W(0,N)_2 \neq 0$ and the first equation implies that $1 - 2p_1 = 1$ i.e. $p_1 = 0$. On the other hand, $\wedge^2(N) \neq 0$ too and the second equation implies that $2 - 2p_1 = 1$ i.e. $p_1 = 1/2$. The contradiction proves that $N \leq 2$. Since $W_0 \simeq K_0$ and $W_1 \simeq K_2$, we concentrate on $W_2 = \mathbb{C}[\partial] \otimes (W(2) \oplus \wedge(2))$. If we apply the second equation to $\xi_1\xi_2$, we get $p_0 = 0$ and $p_1 = 1/2$.

If $R = S_{N,a}$ or $R = \widetilde{S}_N$, then $R = \mathbb{C}[\partial] \otimes W(N)$ as $\mathfrak{sl}_N$-modules. If $N > 2$, $W(N)^2 \neq 0$ and the formulas that appear in the proof of [FK, Proposition 4.16] show that the conformal weight of any vector in $W(N)^2$ is either $(N-2)/N$ or $(2N-2)/N \neq 1$. Therefore $N = 2$ is the only possibility.

If $R = K_N$, $N \geq 5$, the conformal weight of $\xi_1\xi_2\xi_3\xi_4$ with respect to any simple physical Virasoro pair (K_N, L) is 0.

As for K_4', we can check by direct computation that $K'_{4\,\bar{0}} = L \ltimes Cur\ \mathfrak{cso}_4$. Similarly in the case of K_N, $0 \leq N \leq 3$.

Finally, it was proved in [CK] that $CK_{6\,\bar{0}} = L \ltimes Cur\ \mathfrak{so}_6$. □

Remark 6.3. For W_2 we have $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{b}_2$; here $\mathfrak{b}_2$ denotes the two-dimensional solvable Lie algebra. In all other cases, $\mathfrak{g} = \mathfrak{r}$.

6.3. **Physical conformal superalgebras.** A *physical* Lie conformal superalgebra is a simple physical Virasoro pair (R, L) such that the following conditions hold:

(1) $R_{\bar{0}} = L \ltimes \operatorname{Cur} \mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{r} \subseteq \mathfrak{g}$;
(2) $L_{(0)}a = \partial a$ for any $a \in R$.

We thus obtain a generalization of the main result of [K3] (see also [Y]).

Proposition 6.4. *The following is a complete list of physical Lie conformal superalgebras:*

$$\begin{aligned} W_1, &\quad -1 + p_1\partial E \\ W_2, &\quad -1 + \frac{1}{2}\partial E \\ S_{2,0}, &\quad -1 + \frac{1}{2}\partial E \\ K_0, &\quad -1 \\ K_1, &\quad -1 \\ K_3, &\quad -1 \\ K_4', &\quad -1 + (p_0 + p_1\partial)\partial\nu \\ CK_6, &\quad -1 + \alpha\partial^3\nu. \end{aligned}$$

Proof. Condition (2) excludes $\widetilde{S}_2$ from the list in Corollary 6.2, implies that $a = 0$ for $S_{2,a}$ and forces $p_0 = 0$ for (W_1, L). □

The formal distribution Lie superalgebras $(\operatorname{Lie} R, R)$ corresponding to the Lie conformal superalgebras on the list of Proposition 6.4 are well known to physicists (except for CK_6). All of them, except for CK_6, have non-trivial central extensions. In the cases $R = K_0 = W_0, K_1, K_2 = W_1, K_3, S_{2,0}$, and K_4', these central extensions are called the Virasoro, Neveu–Schwarz, $N = 2$, $N = 3$, $N = 4$, and big $N = 4$ superconformal algebras. With the exception of the last one, they appear on the list of [RS]. The last one appeared independently in [KL], [S], and [STP].

7. Representations of Solvable Lie Conformal Superalgebras

In this section we discuss several results towards a possible analog of Lie Theorem for finite Lie conformal superalgebras.

7.1. **Preliminaries.** First we restate the analogs of Lie theorems in the, respectively, non-conformal and conformal non-super cases:

Theorem 7.1. [K2] *Let L be a finite-dimensional solvable Lie superalgebra and V its irreducible representation. Then either* $\dim V_{\bar{0}} = \dim V_{\bar{1}}$ *and* $\dim V = 2^s$*, where* $s \leq \dim L_{\bar{1}}$ *or* $\dim V = 1$.

Theorem 7.2. [DK] *Let R be a finite solvable Lie conformal algebra and M an R-module. Then there exists $v \in M$ such that $x_\lambda v = \varphi(a)v$, $\varphi : R \to \mathbb{C}[\lambda]$. In particular, every non-trivial simple R-module M is free and* $\operatorname{rk} M = 1$.

Remark 7.3. Let R be a Lie conformal algebra with a module M. Then $(\operatorname{Tor} R)_\lambda M = 0$, $\operatorname{Tor} M$ is a submodule of M, and $R_\lambda \operatorname{Tor} M = 0$ [DK]. Since we are interested in simple modules only, below we will always assume that both the conformal algebra and its modules are free over $\mathbb{C}[\partial]$.

We will also need the following

Lemma 7.4. [DK] *For a solvable Lie conformal superalgebra R,* $\operatorname{rk} R > \operatorname{rk} R'$*, where $R' = \langle R_\lambda R \rangle$ is the derived algebra of R.*

Another useful result is the analog of Lemma 1.4:

Lemma 7.5. *A finite module M over a finite Lie conformal superalgebra R either contains a simple R-submodule or a submodule N such that $R_\lambda N = 0$.*

7.2. **Rank 1 modules.** Let R be a finite solvable Lie conformal superalgebra and M an R-module, $\operatorname{rk} M = 1$. In particular, $M = \mathbb{C}[\partial]v$ for some $v \in M$ and the representation of R is given by the function $\ell : R \to \mathbb{C}[\lambda, \partial]$ such that $x_\lambda v = \ell(x)v$ for all $x \in R$. Theorem 7.2 immediately implies that either $\ell = 0$ or M is a simple $R_{\bar{0}}$-module and that ℓ as a function on $R_{\bar{0}}$ depends only on λ. A direct computation shows that $\ell(R'_{\bar{0}}) = 0$. Also, since the action of $R_{\bar{1}}$ must change parity, we conclude that $\ell(R_{\bar{1}}) = 0$.

Conversely, if ℓ satisfies all the above conditions and $\ell(\partial x) = -\lambda\ell(x)$ for all x, ℓ determines an R-module of rank one.

Let $\mathcal{L}$ be the set of functions $\ell : R \to \mathbb{C}[\lambda]$ such that $\ell(R'_{\bar{0}}) = \ell(R_{\bar{1}}) = 0$ and $\ell(\partial x) = -\lambda\ell(x)$ for all $x \in R$. Let $\mathcal{L}_0$ be a subspace of $\mathcal{L}$ consisting of ℓ for which $\ell(R') = 0$.

Remark 7.6. Let H be a subalgebra of R such that $R = H \oplus \mathbb{C}[\partial]x$, $x \in R$, $R_{\bar{0}} \subset H$. The λ-bracket in R induces an H-action on $\mathbb{C}[\partial]x$. The corresponding ℓ lies in $\mathcal{L}_0$.

Let M be an R-module and $\ell \in \mathcal{L}_0$. Let V be a $\mathbb{C}[\partial]$-generating subspace of M, i.e. $M = \mathbb{C}[\partial] \otimes V$. We introduce another R-module structure on M by putting $x_\lambda v = x^M_\lambda v + \ell(x)v$ for $x \in R$, $v \in V$ and extending the R-action to all of M in the standard way. In this way we obtain a module M' and call M and M' *$\mathcal{L}_0$-equivalent*.

7.3. **Induced modules.** In the proof of Theorem 7.1, the powers of 2 appear because, in principle, a simple module V over a Lie superalgebra L can be induced from a smaller module W over a subalgebra H of L. In particular, when $L = H \oplus \mathbb{C}g$ and $p(g) = \bar{1}$, $V = W + gW$, hence the dimension gets doubled.

In the conformal case, non-trivial induction is still possible but the rank can grow arbitrarily as the following lemma demonstrates.

Lemma 7.7. *Let R be a Lie conformal superalgebra, and let H be a (homogeneous) subalgebra of R such that $R' \subset H$ and $R = H \oplus \mathbb{C}[\partial]x$ for some homogeneous $x \in R$. Let M be a finite R-module and N an H-submodule of M that generates M. Then*

$$M = \sum_{i=1}^{l} \mathbb{C}[\partial] \left(\prod_{j=i}^{l} x_{(n_j)} \right) N \text{ for some } n_1, \dots, n_l. \tag{7.1}$$

Moreover, as H-modules M and N have the same simple quotients. If N is simple, then $\operatorname{rk} M$ *is proportional to* $\operatorname{rk} N$.

Proof. Clearly, $\langle x_\lambda N\rangle + N$ is an H-submodule of M and, unless $M = N$, it contains N as a proper submodule.

Hence $U = (\langle x_\lambda N\rangle + N)/N$ is an H-module. Moreover, since $x_{(k)}(h_{(m)}w) = \pm h_{(m)}(x_{(k)}w) \mod N$ for $k, m \in \mathbb{Z}_+$ and $w \in W$, we obtain submodules $U_k = (\mathbb{C}[\partial](x_{(k)}N) + N)/N$ of U. The Leibniz rule implies that $U_k \subset U_{k+1}$. Obviously, $U = \sum U_k$ and it immediately follows that $U = U_n$ for some n. Hence, $N + \langle x_\lambda N\rangle = N + \mathbb{C}[\partial](x_{(n)}N)$. Now we can consider the latter module instead of N. Then (7.1) follows by induction.

We have actually obtained a filtration of M: $N \subseteq N + \mathbb{C}[\partial](x_{(0)}N) \subseteq \ldots \subseteq M$. Denote the k-th element of the filtration by M_k. Then a direct computation shows

that there exists a natural surjective map $N \to M_k/M_{k-1}$ for every k (e.g. $N \to U_k/U_{k-1}$ is defined as $v \mapsto x_{(k)}v$). This proves the rest of the lemma. □

Remark 7.8. By the Jacobi identity, one can always have $n_j \leq n_{j+1}$ in (7.1). When x is odd the inequality is strict. The analogous lemma for ordinary algebras would then imply that $l = 1$ whenever x is odd but this is not true for conformal algebras.

Let H be a subalgebra of R, M a simple R-module and N a simple H-submodule of R such that N generates M over R. Then we say that M is *induced* from N. This does not mean that we can define an induction functor from H-modules to R-modules; the word "induced" is used here only as a shorthand.

Thus Lemma 7.7 provides a description of induced modules when $\operatorname{rk} H = \operatorname{rk} R - 1$ and $R' \subset H$.

Proposition 7.9. *Let M be a finite simple module over a finite solvable Lie conformal superalgebra R. Then all simple factors of M considered as an $R_{\bar{0}}$-module are of rank 1. The corresponding elements of $\mathcal{L}$ extended by zero to $R_{\bar{1}}$ lie in a single coset $\ell_M \in \mathcal{L}/\mathcal{L}_0$.*

Proof. We use induction on $\operatorname{rk} R$.

Let H be a subalgebra of R such that $R = H \oplus \mathbb{C}[\partial]x$, $R' \subset H$, and x is homogeneous. (Such H always exists by Lemma 7.4.) Two cases can occur: M contains a simple H-module N or there exists a submodule $N \subset M$ such that $H_\lambda N = 0$.

Consider the first case, i.e. let M be induced from N. Then by Lemma 7.7 all H-simple factors of M are isomorphic to N. Thus, if x is odd, M and N viewed as $R_{\bar{0}}$-modules have the same simple factors and we are done by induction. However, if x is even, M and N have the same simple factors only as $H_{\bar{0}}$-modules. However, let then ℓ_1, ℓ_2 be the two forms corresponding to the simple factors and let $\ell = \ell_1 - \ell_2$. By induction $\ell(R'_{\bar{1}}) = 0$, hence we are done.

Consider now the second case, i.e. let M be induced from N such that $H_\lambda N = 0$. Then all simple factors of M are killed by H. Hence, if x is odd, M has no $R_{\bar{0}}$-simple factors. If x is even and we get a simple factor that corresponds to $\ell \in \mathcal{L}$, we get $\ell(R'_{\bar{1}}) = 0$ and thus $\ell \in \mathcal{L}_0$. □

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